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Modeling an Electrostatic Elastic Membrane System

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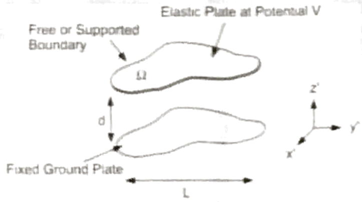
UT Tyler HONORS PROGRAM

Introduction

In recent times, a great interest has been developing with regards to micro- and nano - electromechanical systems (MEMS and NEMS). The purpose of this research was an attempt to understand what goes in to modeling a MEMS system. We chose to model an electrostatic elastic membrane system. This involves an elastic membrane being held at a potential V, over a grounded plate, so as to create a capacitor. As the potential energy increases, the membrane deflects towards the ground plate. However, during experimentation we found that once a certain voltage was reached, the membrane went from being stable with mild deflection, to rapidly deflecting towards the ground plate. We call this voltage the pull in voltage. Our goals are to create a mathematical model and construct a bifurcation diagram for the disk geometry, analyze it using phase plane analysis, check for stability of solutions, and provide an explanation for the pull in voltage.



Energy Equation

In order to derive the model for our system, we first considered the energy acting upon it. We have elastic energy given by the equation

$$T\int_{\Omega}\sqrt{1+\frac{\partial w}{\partial x}^2+\frac{\partial w}{\partial y}^2}-1\,dA$$

and electrostatic energy given by

$$\frac{-\epsilon}{2}\int_{S}|\nabla\phi|^2\,dV$$

As you may notice, electrostatic energy is integrated over a volume, whereas elastic energy is integrated over a surface area. We use the divergence theorem, in the form of Green's identity, to express electrostatic energy in terms of a surface integral. Upon combining our integrals, and assuming a small aspect ratio, we have the force equation

$$\int_{\Omega}T\frac{\epsilon^2}{2}\left(\frac{\partial w^2}{\partial x}+\frac{\partial w^2}{\partial y}\right)+\frac{\epsilon_0 V^2}{dw}\,dA$$

Now we need to minimize the energy in our integral using the Calculus of Variations. For the disk geometry of our membrane, we find our Euler-Lagrange equation to be $\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}=\frac{\lambda}{w^2}$ with $\lambda=\frac{\epsilon_0 V^2 L^2}{2d^3\mu}$.

\with boundary conditions

$$w(1)=1,\text{ and }\frac{dw}{dr}(0)=0$$

Bifurcation Diagram

Notice that currently our model cannot be integrated, and is a boundary valued problem. However, we notice a symmetry, allowing us to scale our equation. If we name our scaling term α, and our independent term $y(\gamma r)$, we have the equation

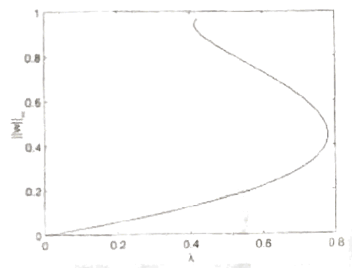
$$w(r)=\alpha y(\omega r)$$

Now we have an initial value problem, given by

$$\frac{d^2y}{dr^2}+\frac{1}{r}\frac{dy}{dr}=\frac{1}{y^2}$$

with the conditions $y(0)=1$, and $\frac{dy}{dr}(0)=0$

We use this to create our bifurcation diagram, shown below.



In order to create our bifurcation diagram, we chose gamma, and calculate the corresponding value in our model. The bifurcation diagram shows there are no solutions once lambda becomes too large, which is consistent with our pull in voltage. However, notice that the top boundary of our diagram is somewhat unresolved. This is due to the nature of our differential equation, because as gamma grows large, the (1/y) aspect of our equation becomes close to zero. We need to be sure the folding near the boundary is not just calculation error, but instead an actual occurrence in our model. As such, we decide to look at it analytically. Currently our model is non autonomous, so we make the following change of variables

$$\eta=\log(r),\ y(r)=r^{\frac{2}{3}}v(\eta)$$

Now we have the following equation

$$\frac{d^2v}{d\eta^2}+\frac{4}{3}\frac{dv}{d\eta}+\frac{4}{9}v-\frac{1}{v^2}=0$$

However, this is difficult to solve as it stands, so we look to make it easier through a change of variables.

Consider: $q=\frac{v'}{v},\ p=\frac{1}{v}.$

Then we have:

$$p'=-pq$$
$$q'=p^3-\left(q+\frac{2}{3}\right)^2.$$

Now we solve for our critical points by setting p' and q' equal to zero. We find two, being:

$$\left(0,-\frac{2}{3}\right)\quad\left(\left(q+\frac{2}{3}\right)^{\frac{2}{3}},0\right)$$

Phase Plane Analysis

In order to conduct phase plane analysis, we must find the Jacobian matrix for our differential equation to analyze the almost linear system. It is given by

$$\begin{pmatrix}-q & -p\\ 3p^2 & 2(q+\frac{2}{3})\end{pmatrix}.$$

Now we plug in the values for our critical points. We find that the eigenvalues for our first and second critical points to be, respectively,

$$\lambda_{1,2}=0,\ \frac{2}{3}\qquad\lambda_{1,2}=-\frac{2}{3}\pm\frac{2\sqrt{2}}{3}i$$

The 0 eigenvalue of our first critical point is somewhat troublesome. However we find its eigenvector negates the p term, making it a function of only q. As such, we can say that we have an unstable node at our first critical point. From the eigenvalues for our second critical point, we can determine that we have a stable spiral at this location.

Notice that previously we could not resolve the top boundary of our bifurcation diagram. Having a stable spiral at that point would provide some explanation as to the folding behavior of our diagram, and approaching this critical point gives lambda approaching 4/9 and 1-w(0) approaching 1 which is consistent with our previous calculations.

Stability

We are interested in stable solutions for our model. Since we are using the disk geometry, we use the equation:

$$-w_t+w_{rr}+\frac{1}{r}w_r=\frac{\lambda}{(1+w)^2},$$

where $-w_t$ represents our time variable. We know that

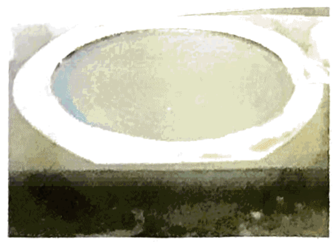
$$w(r,t)=w^*(r)+\epsilon e^{kt}v(r)+O(\epsilon^2)$$

Using these two relationships, we can Taylor expand the right side of our first equation about epsilon equal to zero to find

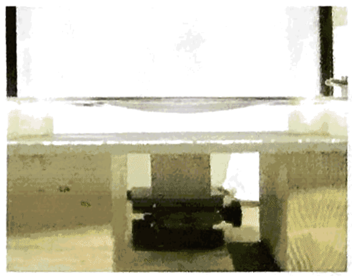
$$v_{rr}+\frac{1}{r}v_r+\frac{2\lambda v}{(1+w^*)^3}=kv$$

We use this differential equation for v to make an eigenvalue equation for k. We know the form of second and first derivative matrixes, and use these to create a matrix problem Av = kv. Now we solve for eigenvalues. For any value of k greater than zero, we are unstable. This is due to the fact that as t approaches infinity, e^{kt} will approach infinity for positive k. So, we know a point is stable if k is less than zero. On our bifurcation diagram, this is represented by the values before the first fold. These are our stable solutions.

Experiments



We conducted experiments in the school laboratory to test our model. Above and below is a picture of our setup. The top hole was filled with a thin soap layer, and was raised to different potentials. The bottom layer was grounded, and created a capacitor. We found that our model accurately represented our experimental results. After reaching our pull in voltage corresponding to roughly lambda = .78, we found no stable solutions. This is consistent with our model.



Conclusion

We began by using the forces acting upon our system to derive the model. For the disk geometry of our model, we calculated the bifurcation diagram. This was analyzed using phase plane analysis, and this reassured us of our original diagram. We then continued on to discuss stability of the system, and found that solutions corresponding to the bottom branch of the bifurcation curve are stable, all others are unstable. This provided an explanation for the pull in voltage. Finally, we tested our model through laboratory experiments, and found that they matched our model.

REFERENCES

1. D. H. Bernstein, J. A. Pelesko, *Modeling MEMS and NEMS*, CRC Press, 2002.
2. T.Y. Na, *Computational Methods in Engineering: Boundary Value Problems*, Academic Press, 1979.