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GROUP ACTIONS ON HYPERSPACES

by

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Contents

Abstract

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In this thesis, we will look at the structure of two spaces associated with a topological space X, $\mathscr{C}(X)$ and $\mathcal{P}(X)$. Furthermore, from the group action of a topological group G on X, we get the induced group action of G on $\mathscr{C}(X)$ and $\mathcal{P}(X)$. We will also look at few properties for actions of G on a compact Hausdorff space X: proximality, strong proximality and extreme proximality followed by the main result to give parallel characterizations of proximality and extreme proximality.

Chapter 1

Introduction

Topological dynamics is the study of topological group actions on topological spaces. Often, it yields interesting insights into the acting topological group which are beyond than that can be attained through a direct study of the group itself. A very common way of attaining such insights is to pass from a (continuous) action of a topological group G on a topological space X to an action of G on a space closely associated to, but with more properties, than the original space X.In the context of this paper, it turns out that an action $G \curvearrowright^{\alpha} X$ on a compact Hausdorff space X induces an action of G on $\mathcal{P}(X)$, the space of Radon probability measures on X and $\mathscr{C}(X)$, the space of nonempty closed subsets of X.

Apart from being compact Hausdorff spaces, $\mathcal{P}(X)$ and $\mathscr{C}(X)$ have an extra structure: $\mathcal{P}(X)$ has a natural convex structure (that is, for $\mu, \nu \in \mathcal{P}(X)$ and $t \in [0,1]$, there is a canonical way to define a measure $t\mu + (1-t)\nu \in \mathcal{P}(X)$) and $\mathscr{C}(X)$ has the structure of a join semilattice (where the partial order is given by set inclusion). In fact, as we will see, for $C_1, C_2 \in \mathscr{C}(X)$, one can view the union operation $C_1 \cup C_2$ as a kind of "nontrivial convex combination" of C_1, C_2, and under this, some main results from the theory of compact convex subsets of a locally convex topological vector space (namely, the Krein-Milman theorem and Milman's converse) have analogues in $\mathscr{C}(X)$.

In this thesis, we will look at a few properties for actions of a topological group G on a compact Hausdorff space X: proximality, strong proximality, and extreme proximality. Our main result will be to give parallel characterizations of proximality and extreme proximality. In Chapter 2 we begin by providing some background information about the structures of $\mathcal{P}(X)$ and $\mathscr{C}(X)$. Furthermore, in Chapter 3, we talk about the group action of topological group on the spaces associated with compact Hausdorff space X, followed by our main result.

Chapter 2

Two spaces associated with a topological space

2.1 Preliminaries

See [6, Section 2.5] for more information about convex subsets of a locally convex topological vector space, the proof of the Krein-Milman theorem, and the proof that $\text{ext}(\mathcal{P}(X)) = \{\delta_x : x \in X\}$. Throughout this paper, the field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition. Suppose B is a vector space over $\mathbb{F}$ and $A \subseteq B$. A is said to be a **convex subset** of B, if for every $a_1, a_2 \in A$ and any $p \in (0,1)$ we get $pa_1 + (1-p)a_2 \in A$.

Definition. A **topological vector space** X over the topological field $\mathbb{F}$ is a vector space equipped with a Hausdorff topology such that the vector addition $+\colon X \times X \to X$ and the scalar multiplication : $\mathbb{F} \times X \to X$ are continuous functions.

Note 2.1. The domains of the vector addition and scalar multiplication functions have the product topologies.

Definition. Let (X, τ) be a topological space. If $N \subseteq X$ and $x \in X$, we say that N is a **neighborhood** of x if there is some $U \in \tau$ i.e. U is an open set in X, such that $x \in U \subseteq N$.

Note 2.2. An open set is a neighborhood of every element of itself.

Definition. Let (X, τ) be a topological space. For each $x \in X$, let $\mathscr{F}_x$ be the collection of neighborhoods of x; we call $\mathscr{F}_x$ the **neighborhood filter** of x.

Definition. A topological vector space X is said to be **locally convex**, if for all $x \in X$ and $N \in \mathscr{F}_x$, there is a convex set $C \in \mathscr{F}_x$ such that $C \subseteq N$.

Definition. Let X be a convex subset of a vector space V over $\mathbb{F}$, then $x \in X$ is said to be an **extreme point** of X, if there does not exist $x_1, x_2 \in X$ and $0 < p < 1$ such that $x_1 \neq x_2$ and $x = px_1 + (1-p)x_2$. In other words, an **extreme point** of X is a point which does not lie on the line segment between any two distinct points of X. The set of extreme points of X is denoted by $\textbf{ext}(X)$.

Definition. **Convex hull** of a set A, is the smallest convex set that contains A. It is denoted as $\text{co}(A)$. **Closed Convex Hull** is the closure of the convex hull.

Lemma 2.3. If X is a topological vector space over $\mathbb{F}$ and $A \subseteq X$, then $\overline{co}(A)$ is convex.

(Proof) Let $a_1, a_2 \in \overline{co}(A)$ and $t \in (0,1)$. We need to show that $ta_1 + (1-t)a_2 \in \overline{co}(A)$. Let U be an open set such that $ta_1 + (1-t)a_2 \in U$, which gives $a_1 \in t^{-1}(U - (1-t)a_2)$; where $aU + b = \{au + b : u \in U, b \in X \text{ and } a \in \mathbb{F}\}$ and $aU + b$ is an open set.
Since $a_1 \in \overline{co}(A)$, then there is $x_1 \in co(A)$ such that $x_1 \in t^{-1}(U - (1-t)a_2)$ so that $tx_1 + (1-t)a_2 \in U$ which implies $a_2 \in (1-t)^{-1}(U - tx_1)$.
Since $a_2 \in \overline{co}(A)$, then there is $x_2 \in co(A)$ such that $x_2 \in (1-t)^{-1}(U - tx_1)$ which implies $tx_1 + (1-t)x_2 \in U$, where $tx_1 + (1-t)x_2 \in co(A)$. Hence $\overline{co}(A)$ is convex. □

Theorem 2.4 (Krein Milman Theorem)**.** A compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points.

Theorem 2.5 (Milman Theorem)**.** Let X be a locally convex topological vector space and K be a compact subset of X. If closed convex hull $\overline{co}(K)$ of K is also compact, then K contains all $\text{ext}(\overline{co}(K))$.

2.2 Probability Measures on X

In this section, we will be defining our first space associated with topological space X, which is $\mathcal{P}(X)$. We will define $\mathcal{P}(X)$ in two different ways, first as the set of Radon Borel probability measures on X and second as the subset of $(\mathcal{C}(X))^*$, where $\mathcal{C}(X)$ is the space of real valued continuous functions on X and $(\mathcal{C}(X))^*$ is the dual space of $\mathcal{C}(X)$.

Definition. Two points x and y in a topological space X are said to be **neighborhood-separable** if there exist a neighborhood U of x and a neighborhood V of y such that U and V are disjoint.

Definition. A topological space X is called **Hausdorff space**, if all distinct points in X are pairwise neighborhood-separable.

Definition. A collection $\mathscr{S}$ of subsets of X is called a **σ-algebra** if it contains X, is closed under complement, is closed under countable unions. It follows that σ-algebra is closed under countable intersections.

Definition. For a topological space X, the collection $\mathscr{B}$ of **Borel sets** is defined as the smallest σ-algebra of subsets of X that contains all open sets of X (equivalently, contains all the closed sets of X).

Definition. The measure μ is called **outer regular** if, for any Borel set B, $\mu(B)$ is the infimum of $\mu(O)$ over all open sets O of X containing B.

Definition. The measure μ is called **inner regular on open sets** if, for any open set O, $\mu(O)$ is the supremum of $\mu(C)$ over all compact sets C of X contained in O.

Definition. A **Radon Borel probability measure** on X is a function
$\mu \colon \mathscr{B} \to [0,1]$ that is finite on all compact sets, inner regular on open sets and outer regular and $\mu(X) = 1$

Definition. Let X, Y and V be vector spaces over the same field $\mathbb{F}$. A **bilinear map** is a function $b \colon X \times Y \to V$ such that for all $x \in X$, the map $y \to b(x,y)$ is a linear map from Y to V, and for all $y \in Y$, the map $x \to b(x,y)$ is a linear map from X to V.
Notation: $y \to b(x,y)$ is denoted by b(x , **.**)$\colon Y \to V$ and $x \to b(x,y)$ is denoted by b(**.** , y)$\colon X \to V$.

Definition. Let X and Y be vector spaces over the topological field $\mathbb{F}$ and $b\colon X \times Y \to \mathbb{F}$ be a bilinear map. The **weak topology** on X induced by Y and b, is the weakest topology on X, which make all the maps $\mathrm{b}(\,.\,,y)\colon X \to \mathbb{F}$ continuous for all $y \in Y$.

Definition. Let X be a normed vector space over the field $\mathbb{F}$ and X^* be the set of continuous linear functionals on X. Now X^* is a vector space over the same field $\mathbb{F}$. There exists a bilinear map $\mathrm{b}(\,.\,,\,.)\colon X \times X^* \to \mathbb{F}$ defined as $b(x,\phi) = \phi(x)$, for $x \in X, \phi \in X^*$.
The **weak* topology** on X^* is defined as the weak topology induced by X and b as defined above.

Suppose X is a compact Hausdorff topological space.

Definition. $\mathcal{P}(X)$ is the set of all Radon Borel probability measures on X. It is a compact Hausdorff space under weak * topology, which is induced on it as a subset of $(\mathcal{C}(X))^*$.

Define $\phi : \mathcal{P}(X) \to (\mathcal{C}(X))^*$ such that $\phi(\mu)(f) = \int f d\mu$ where $\phi(\mu) \in (\mathcal{C}(X))^*$ for $\mu \in \mathcal{P}(X)$
Since the map ϕ is one-one, then $\mathcal{P}(X) \subseteq (\mathcal{C}(X))^*$.

Then the topology on the $\mathcal{P}(X)$ is the weak * topology induced from the $(\mathcal{C}(X))^*$, defined as below.
Let (μ_λ) be a net in $\mathcal{P}(X)$. Then the net (μ_λ) converges in the weak * topology to $\mu \in \mathcal{P}(X)$, if $b(f, \phi(\mu_\lambda))$ converges to $b(f, \phi(\mu))$ for all $f \in \mathcal{C}(X)$, where $b\colon \mathcal{C}(X) \times (\phi(\mathcal{P}(X))) \to \mathbb{R}$ is a bilinear form defined as $b(f, \phi(\mu)) = \phi(\mu)(f) = \int f d\mu$.

noted : $\phi(\mu)$ is the image of μ in $(\mathcal{C}(X))^*$.

Lemma 2.6. $\mathrm{ext}(\mathcal{P}(X)) = \{\delta_x : x \in X\}$, where

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Therefore by Krein Milman theorem applied to $\mathcal{P}(X)$, we get that $\mathcal{P}(X) = \overline{co}(\{\delta_x ; x \in X\})$

2.3 Closed subsets of X

In this section, we will be defining the second space associated with topological space X, which is $\mathscr{C}(X)$. See [5] for more information about $\mathscr{C}(X)$.

Definition. $\mathscr{C}(X)$ is the collection of nonempty closed subsets of X. It is a compact Hausdorff topological space under the Vietoris topology.

This topology is generated by subsets of $\mathscr{C}(X)$ as given below.

$$\langle U_1, \ldots, U_n \rangle := \{E \in \mathscr{C}(X) : E \subseteq \cup_{i=1}^{n} U_i \text{ and } E \cap U_i \neq \emptyset \ \forall i\}$$

where $U_1, \ldots, U_n$ are open subsets of X, and such a collection form the topology on $\mathscr{C}(X)$

noted: For the next lemma, the nontrivial convex combinations in $\mathscr{C}(X)$ are defined by $tC_1 + (1-t)C_2 = C_1 \cup C_2$, for $0 < t < 1$ and $C_1, C_2 \in \mathscr{C}(X)$. Indeed the singletons are the extreme points.
Although, $\mathscr{C}(X)$ is not a compact convex subset of a locally convex topological space, even then we get the analogous of Krein Milman theorem in $\mathscr{C}(X)$.

Lemma 2.7. $\mathscr{C}(X) = \overline{co}(\{\{x\} : x \in X\})$

(Proof) Let $C \in \mathscr{C}(X)$ and $\langle U_1, \ldots, U_n \rangle$ be any neighborhood of C, then
$C \subseteq \bigcup_{i=1}^{n} U_i$ and $C \cap U_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. Choose $x_i \in C \cap U_i$ so that $\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} U_i$ and $\{x_1, \ldots, x_n\} \cap U_i \neq 0$ for all $i \in \{1, \ldots, n\}$ which gives that $\{x_1, \ldots, x_n\} \in \langle U_1, \ldots, U_n \rangle$, where $\{x_1, \ldots, x_n\} \in co\{\{x\}; x \in X\}$. Therefore $co\{\{x\}; x \in X\}$ is dense in $\mathscr{C}(X)$.
Since the set $co(\{\{x\} : x \in X\})$ is dense in $\mathscr{C}(X)$, we get that $\overline{co}(\{\{x\} : x \in X\}) = \mathscr{C}(X)$. $\square$

Theorem 2.8 (Milman Theorem in the context of $\mathscr{C}(X)$)**.** If $\mathcal{F} \subseteq \mathscr{C}(X)$ satisfies $\overline{co}(\mathcal{F}) = \mathscr{C}(X)$, then $\{\{x\}\colon x \in X\} \subseteq \overline{\mathcal{F}}$.

Chapter 3

Group Actions

See [4] for much more information about proximal, strongly proximal, and extremely proximal actions.
Suppose X is a compact Hausdorff space and G be a topological group, that is a group such that the group operations $G \to G$, $g \mapsto g^{-1}$; and $G \times G \to G$, $(g, h) \mapsto gh$, are continuous.

Definition. We denote the set of homeomorphisms from X to X by $\mathrm{Homeo}(X)$. Note that this is a group under the usual operations of function inverse and composition.

Definition. A **group action** of G on X, denoted as $G \curvearrowright^{\alpha} X$, is defined to be a group homomorphism $\alpha\colon G \to \mathrm{Homeo}(X)$ such that $G \times X \to X$, $(g, x) \mapsto \alpha(g)(x)$ is continuous. We will usually write either $\alpha_g(x)$ or $g.x$ for $\alpha(g)(x)$. Note that since α is a homomorphism, it satisfies the following three conditions:

1. $\alpha_{g_1 g_2}(x) = \alpha_{g_1}(\alpha_{g_2}(x))$; for all $g_1, g_2 \in G$ and for all $x \in X$.
2. $\alpha_e(x) = x$; for all $x \in X$, where e is the identity element of G.
3. For all $g \in G$ and $x \in X$, $\alpha_{g^{-1}}(x) = \alpha_g^{-1}(x)$.

For $x \in X$, we define the **orbit** of x to be the set $G.x = \{g.x : g \in G\}$.

Definition. A group action $G \curvearrowright^{\alpha} X$ is called **minimal**, if for all $x \in X$, the orbit $G.x$ is dense in X.

Proposition 3.1. A group action $G \curvearrowright^{\alpha} X$ is minimal if and only if there is no non empty proper, closed and G-invariant subset of X.

(Proof) Suppose $G \curvearrowright^{\alpha} X$ is minimal. Let $A \subseteq X$ be a nonempty proper, closed and G-invariant subset of X. Let $x \in X$ and $a \in A$. Then $x \in \overline{G.a} = X$. Since A is G-invariant subset of X, then $G.a \subseteq A$. Also, since A is closed subset of X, then $\overline{G.a} \subseteq \overline{A} = A$ i.e. $x \in A$. Therefore $X = A$.
Conversely, suppose there is no non empty proper closed G-invariant subset of X. Let $x \in X$. Now $\overline{G.x}$ is a nonempty closed and G-invariant subset of X. Therefore $\overline{G.x} = X$. $\square$

Definition. For an action $G \curvearrowright^{\alpha} X$, we make the following definitions:

- For $C \subseteq X$ and $g \in G$, $g.C = \{g.c : c \in C\}$.
- For $\mu \in \mathcal{P}(X)$ and $g \in G$, $(g.\mu)(B) = \mu(g^{-1}.B)$; for all $B \in \mathscr{B}$.

The following proof is taken from the proof of [1, Theorem 2].

Lemma 3.2. If $G \curvearrowright X$ is a continuous action of a topological group G on a topological space X, then for each compact $C \subseteq X$ and open $V \subseteq X$, the set

$$\{s \in G : sC \subseteq V\}$$

is open in G.

(Proof) Suppose g is in the set described above. Then, by continuity of the map $G \times X \to X$, $(g, x) \mapsto g.x$, for any $x \in C$, there exist an open neighborhood V_x of x and an open neighborhood $U_{g,x}$ of g such that $U_{g,x}V_x \subseteq V$ (where $U_{g,x}V_x = \{sy : s \in U_{g,x}, y \in V_x\}$). Now, $C \subseteq \cup_{x \in C} V_x$. Since C is compact set, then, there exists $x_1, \dots, x_n$ such that

$$C \subseteq V_{x_1} \cup \cdots \cup V_{x_n}.$$

Set $U_g = U_{g,x_1} \cap \cdots \cap U_{g,x_n}$. Now U_g is an open neighborhood of g. We need to show that $U_g \subseteq \{s \in G : sC \subseteq V\}$. Let $s' \in U_g$, then for any $x \in C$, $x \in V_{x_k}$, for some k and $s' \in U_{g,x_k}$, $s'x \in U_{g,x_k}V_{x_k}$. Therefore $s'C \subseteq V$. $\square$

For the following, let $G \curvearrowright X$ be a continuous action of a topological group G on a compact Hausdorff space X. Denote by $\varphi : G \times \mathscr{C}(X) \to \mathscr{C}(X)$ the map $\varphi(g, C) = gC$.

The next two lemmas are essentially taken from [2, Remarks 4.1, 4.4, 4.6].

Lemma 3.3. For every open $U \subseteq X$, the set $\varphi^{-1}(\langle U \rangle)$ is open in $G \times \mathscr{C}(X)$.

(Proof) Let $U \subseteq X$ be open, and set $(g, C) \in \varphi^{-1}(\langle U \rangle)$. We need to find an open neighborhood of (g, C) that is contained in $\varphi^{-1}(\langle U \rangle)$.

Since $gC \subseteq U$, we have $C \subseteq g^{-1}U$, so that $X \setminus g^{-1}U \subseteq X \setminus C$. Since X is locally compact, there exists an open set $V \subseteq X$ such that

$$X \setminus g^{-1}U \subseteq V \subseteq \overline{V} \subseteq X \setminus C.$$

Then

$$C \subseteq X \setminus \overline{V} \subseteq X \setminus V \subseteq g^{-1}U,$$

which implies

$$gC \subseteq g(X \setminus \overline{V}) \subseteq g(X \setminus V) \subseteq U.$$

Let $W = \{g \in G : g(X \setminus V) \subseteq U\}$, which is open by Lemma 3.2, and consider the set

$$W \times \langle X \setminus \overline{V} \rangle \subseteq G \times \mathscr{C}(X).$$

This is an open neighborhood of (g, C).

Let $(h, D) \in W \times \langle X \setminus \overline{V} \rangle$, then

$$\varphi(h, D) = hD \subseteq h(X \setminus \overline{V}) \subseteq h(X \setminus V) \subseteq U.$$

Thus, $W \times \langle X \setminus \overline{V} \rangle \subseteq \varphi^{-1}(\langle U \rangle)$.

$\square$

For a subset $S \subseteq X$, make the following notation:

$$[S] = \{C \in \mathscr{C}(X) : C \cap S \neq \emptyset\}.$$

Note that if $U \subseteq X$ is open, then $[U]$ is open in $\mathscr{C}(X)$, since $[U] = \langle U, X \rangle$.

Lemma 3.4. For every open $U \subseteq X$, the set $\varphi^{-1}([U])$ is open in $G \times \mathscr{C}(X)$.

(Proof) Let $U \subseteq X$ be open, and take $(g, C) \in \varphi^{-1}([U])$. We need to find an open neighborhood of (g, C) that is contained in $\varphi^{-1}([U])$.

Then by the definition of $\varphi^{-1}([U])$ and $[U]$, $gC \cap U \neq \emptyset$. So there exists an $x \in C$ such that $gx \in U$. By the continuity of the map $G \times X \to X$, $(s, y) \mapsto s.y$, there exist an open neighborhood W of g and open neighborhood V of x such that $WV \subseteq U$ (where $WV = \{s.y : s \in W, y \in V\}$). Consider the set

$$W \times [V] \subseteq G \times \mathscr{C}(X).$$

This is an open neighborhood of (g, C). Let $(h, D) \in W \times [V]$, then there is a point $y \in D \cap V$, so that $hy \in U$. Hence $hD \cap U \neq \emptyset$, i.e., $(h, D) \in \varphi^{-1}([U])$. Thus, $W \times [V] \subseteq \varphi^{-1}([U])$.

□

Proposition 3.5. If $G \curvearrowright^{\alpha} X$ is an action, then the map $\tilde{\alpha} : G \to \mathrm{Homeo}(\mathscr{C}(X))$, $\tilde{\alpha}(g)(C) = g.C$ is an action on $\mathscr{C}(X)$.

(Proof) Let g_1, $g_2 \in G$ and $C \in \mathscr{C}(X)$. Then $(\tilde{\alpha}(g_1)\tilde{\alpha}(g_2))(C) = (\tilde{\alpha}(g_1))(\tilde{\alpha}(g_2)(C)) = \tilde{\alpha}(g_1)(g_2.C) = g_1.(g_2.C) = (g_1g_2).C = \tilde{\alpha}(g_1g_2)(C)$. Therefore, $\tilde{\alpha}(g_1)\tilde{\alpha}(g_2) = \tilde{\alpha}(g_1g_2)$. So $\tilde{\alpha}$ is a group homomorphism.
Let $C \in \mathscr{C}(X)$ and $(g.x_i) \to y$, for a net (x_i) in C. Then $x_i = g^{-1}(g.x_i) \to g^{-1}.y$. Since $g^{-1}.y \in C$, then $y \in g.C$. Therefore $g.C \in \mathscr{C}(X)$.
Let $\tilde{\alpha}(g)(C_1) = \tilde{\alpha}(g)(C_2)$, then $g.C_1 = g.C_2$, which can be written as $g^{-1}g.C_1 = g^{-1}g.C_2$, so that $C_1 = C_2$. Therefore $\tilde{\alpha}(g)$ is injective. Now for any $C \in \mathscr{C}(X)$ we have $g^{-1}.C \in \mathscr{C}(X)$ such that $\tilde{\alpha}(g)(g^{-1}.C) = C$, Therefore $\tilde{\alpha}(g)$ is surjective.
Let $\langle U_1, \ldots, U_n \rangle \subseteq \mathscr{C}(X)$ be any basic open set in $\mathscr{C}(X)$. Now $(\tilde{\alpha}(g))^{-1}(\langle U_1, \ldots, U_n \rangle) = \{C \in \mathscr{C}(X) : \tilde{\alpha}(g)(C) \in \langle U_1, \ldots, U_n \rangle\} = \{C \in \mathscr{C}(X) : g.C \in \langle U_1, \ldots, U_n \rangle\} = \{C \in \mathscr{C}(X) : C \in \langle g^{-1}U_1, \ldots, g^{-1}U_n \rangle\}$. So, $\tilde{\alpha}(g))^{-1}(\langle U_1, \ldots, U_n \rangle) = \langle g^{-1}U_1, \ldots, g^{-1}U_n \rangle$, which is open in $\mathscr{C}(X)$. Therefore $\tilde{\alpha}(g)$ is continuous.

Let $\langle U_1, \ldots, U_n \rangle$ be a basic open set in $\mathscr{C}(X)$ (for open sets $U_1, \ldots, U_n \subseteq X$). Note that

$$\langle U_1, \ldots, U_n \rangle = \langle \cup_{k=1}^{n} U_k \rangle \cap [U_1] \cap \cdots \cap [U_n],$$

so that

$$\varphi^{-1}(\langle U_1, \ldots, U_n \rangle) = \varphi^{-1}(\langle \cup_{k=1}^{n} U_k \rangle) \cap \varphi^{-1}([U_1]) \cap \cdots \cap \varphi^{-1}([U_n]),$$

which is open by Lemmas 3.3 and 3.4.

□

Lemma 3.6. $G \curvearrowright^{\alpha} X$ gives $G \curvearrowright C(X)$ such that for all $f \in C(X)$, $g \to g.f$ is norm continuous, where $C(X)$ is the set of continuous functions from X to X.

Proposition 3.7. If $G \curvearrowright^{\alpha} X$ is an action, then the map $\tilde{\tilde{\alpha}} : G \to \mathrm{Homeo}(\mathcal{P}(X))$, $\tilde{\tilde{\alpha}}(g)(\mu) = g.\mu$ is an action on $\mathcal{P}(X)$

(Proof) Let $g_1,\ g_2 \in G$ and $\mu \in \mathcal{P}(X)$. Then $(\tilde{\tilde{\alpha}}(g_1)\tilde{\tilde{\alpha}}(g_2))(\mu) = (\tilde{\tilde{\alpha}}(g_1))(\tilde{\tilde{\alpha}}(g_2(\mu)) = \tilde{\tilde{\alpha}}(g_1)(g_2.\mu) = g_1.(g_2.\mu) = (g_1g_2).\mu = \tilde{\tilde{\alpha}}(g_1g_2)(\mu)$. Therefore, $(\tilde{\tilde{\alpha}}(g_1)\tilde{\tilde{\alpha}}(g_2)) = \tilde{\tilde{\alpha}}(g_1g_2)$. So $\tilde{\tilde{\alpha}}$ is a group homomorphism.
Let $g_i \to g$ and $\mu_i \to \mu$. Fix $f \in C(X)$. We need to show that $|\langle g_i\mu_i - g\mu, f\rangle| \to 0$. By Lemma 3.6 and $g_i \to g$, we get i_1 such that $||g_i^{-1}f - g^{-1}f|| < \frac{\epsilon}{2}$ for all $i \geq i_1$. Choose i_2 such that $|\langle \mu_i - \mu, g^{-1}f\rangle| < \frac{\epsilon}{2}$ for all $i \geq i_2$. Then for $i \geq max\{i_1, i_2\}|\langle g_i\mu_i - g\mu, f\rangle| \leq |\langle g_i\mu_i - g\mu_i, f\rangle| + |\langle g\mu_i - g\mu, f\rangle| = |\langle \mu_i, g_i^{-1}f - g^{-1}f\rangle| + |\langle \mu_i - \mu, g^{-1}f\rangle| \leq ||g_i^{-1}f - g^{-1}f|| + \frac{\epsilon}{2} < \epsilon$.
Therefore the map $G \times \mathcal{P}(X) \to \mathcal{P}(X)$ is continuous.
Let $\tilde{\tilde{\alpha}}(g)(\mu_1) = \tilde{\tilde{\alpha}}(g)(\mu_2)$, then $g.\mu_1 = g.\mu_2$, which can be written as $g^{-1}g.\mu_1 = g^{-1}g.\mu_2$, so that $\mu_1 = \mu_2$. Therefore $\tilde{\tilde{\alpha}}(g)$ is injective. Now for any $\mu \in \mathcal{P}(X)$ we have $g^{-1}.\mu \in \mathcal{P}(X)$ such that $\tilde{\tilde{\alpha}}(g)(g^{-1}.\mu) = \mu$, Therefore $\tilde{\tilde{\alpha}}(g)$ is surjective.
Let $\mu_i \to \mu$ in $\mathcal{P}(X)$, then for $g \in G$, we have $g.\mu_i \to g.\mu$ i.e. $\tilde{\tilde{\alpha}}(g)(\mu_i) \to \tilde{\tilde{\alpha}}(g)(\mu)$. Hence the map $\tilde{\tilde{\alpha}}(g)$ is continuous.

$\square$

Definition. A group action $G \curvearrowright^{\alpha} X$ is called **proximal**, if for any two points x, y in X, there is a net (g_i) in G such that $\lim_i g_i.x = \lim_i g_i.y$

Definition. A group action $G \curvearrowright^{\alpha} X$ is called **extremely proximal**, if with respect to the corresponding induced group action $G \curvearrowright^{\tilde{\alpha}} \mathscr{C}(X)$, for all $C \in \mathscr{C}(X)$ and for all $y \in X$, there exists a net (g_i) in G such that $\lim_i g_i.C$ is a singleton.

Definition. A group action $G \curvearrowright^{\alpha} X$ is called **strongly proximal**, if for all $\mu \in \mathcal{P}(X)$, there exists a net (g_i) in G such that $g_i.\mu \to \delta_x$ for some $x \in X$.

Proposition 3.8. If $G \curvearrowright^{\alpha} X$ is strongly proximal action then it is proximal action.

(Proof) Suppose $G \curvearrowright^{\alpha} X$ is a strongly proximal action. Let $x,\ y \in X$. Since $G \curvearrowright^{\alpha} X$ is strongly proximal, then there exists a net (g_i) in G such that $g_i.\mu \to \delta_z$ for some $z \in X$ where $\mu = \frac{1}{2}(\delta_x) + \frac{1}{2}(\delta_y)$. By the compactness of X, we can assume that $g_i.x \to x_0$ in X and $g_i.y \to y_0$ in X. Therefore, $\delta_{g_i.x} \to \delta_{x_0}$ and $\delta_{g_i.y} \to \delta_{y_0}$. Now, consider $\mu_i = g_i.\mu = \frac{1}{2}(\delta_{g_i.x}) + \frac{1}{2}(\delta_{g_i.y}) \to \frac{1}{2}(\delta_{x_0}) + \frac{1}{2}(\delta_{y_0})$. So, $\delta_z = \frac{1}{2}(\delta_{x_0}) + \frac{1}{2}(\delta_{y_0})$. Since $\delta_z \in ext(\mathcal{P}(X))$, then $\delta_z = \delta_{x_0} = \delta_{y_0}$. Therefore $x_0 = y_0 = z$. Hence $lim_i\ g_i.x = lim_i\ g_i.y$. $\square$

The following proposition was proved in [3, Theorem 2.3] (but the numbering in this paper has a mistake, and this result should be Theorem 3.3).

Proposition 3.9. If $G \curvearrowright^{\alpha} X$ is extremely proximal then it is strongly proximal in the case that X doesn't have exactly two points (in which case minimal actions are always extremely proximal and never strongly proximal).

Now let's establish few results which will be used to prove the parallel characteristics of proximal and extremely proximal for minimal actions.

Lemma 3.10. If $s_ix_j \to y_j$ for all $j \in \{1, \ldots, n\}$ then $\{s_ix_1, \ldots, s_ix_n\} \to \{y_1, \ldots, y_n\}$ in $\mathscr{C}(X)$.

(Proof) Let U be a basis set of Vietoris topology on $\mathscr{C}(X)$ then $U = \langle U_1, \ldots, U_m\rangle$ for open sets U_k in X.
Now $\{y_1, \ldots, y_n\} \in \langle U_1, \ldots, U_m\rangle$.

Fix $y_j \in \{y_1, \dots, y_n\}$. Now by definition we get open sets $U_1^j, \dots, U_l^j$ where $1 \leq l \leq n$ such that $y_j \in U_q^j$ for $q \in \{1, \dots, l\}$.
Since $s_i x_j \to y_j$, then for U_q^j there is $N_q^j \in \mathbb{N}$ such that $s_k x_j \in U_q^j$ for all $p \geq N_q^j$ where $q \in \{1, \dots, l\}$.
Choose $N^j = max\{N_1^j, \dots, N_p^j\}$ then $s_p x_j \in U_q^j$ for all $p \geq N^j$
where $q \in \{1, \dots, l\}$. This holds for all $y_j \in \{y_1, \dots, y_n\}$.
Choose $N = max\{N^1, \dots, N^n\}$ such that $s_k xj \in U_q^j$ for all $p \geq N$ and all $j \in \{1, \dots, n\}$ where $q \in \{1, \dots, l\}$ i.e. $\{s_p x_1, \dots, s_p x_n\} \in \langle U_1, \dots, U_m \rangle$ for all $p \geq N$. □

Lemma 3.11. Suppose $G \curvearrowright^{\alpha} X$ is minimal. If $K \subseteq \mathscr{F}(X)$ is a G-invariant, convex, relatively closed and contains any singleton set then $K = \mathscr{F}(X)$.

(Proof) Suppose $\{x_0\} \in K$ for $x_0 \in X$. Let $x \in X$ such that $x \neq x_0$. Now the minimality of $G \curvearrowright^{\alpha} X$ gives $(g_i)_{i \in I} \in G$ such that $(g_i.x_0)_{i \in I} \to x$.
Therefore $g_i.\{x_0\} = \{g_i.x_0\} \to \{x\}$. Since K is G-invariant and relatively closed, then $\{x\} \in K$.
Therefore K contains $\{\{x\}; x \in X\}$. Let $A \in \mathscr{F}(X)$, then $A = \bigcup_{x \in A} \{x\}$.
Since K is convex, then $A \in K$. Hence $K = \mathscr{F}(X)$. □

Lemma 3.12. Suppose $G \curvearrowright^{\alpha} X$ is minimal
$G \curvearrowright^{\alpha} X$ is extremely proximal iff $\{\{x\}; x \in X\} \subseteq \overline{G.C}$ for all $C \in \mathscr{C}(X) \setminus \{X\}$.

(Proof) Suppose $G \curvearrowright^{\alpha} X$ is an extremely proximal. Let $C \in \mathscr{C}(X)$, then there exists some $x \in X$ such that $\{x\} \in \overline{G.C}$, i.e. $G.\{x\} \in \overline{G.C}$ so that $\overline{G.\{x\}} \subseteq \overline{G.C}$. Let $y \in X$ such that $y \neq x$. By the minimality of $G \curvearrowright^{\alpha} X$, we get $y \in \overline{G.x}$; for all $y \in X$, so that $\{y\} \in \overline{G.\{x\}} \subseteq \overline{G.C}$.
Therefore, $\{\{x\}; x \in X\} \subseteq \overline{G.C}$; for all $C \in \mathscr{C}(X) \setminus \{X\}$.
Conversely, suppose $\{\{x\}; x \in X\} \subseteq \overline{G.C}$ for all $C \in \mathscr{C}(X) \setminus \{X\}$. It follows trivially that for all $C \in \mathscr{C}(X)$ and for all $x \in X$, $\{x\} \in \overline{G.C}$. Hence $G \curvearrowright^{\alpha} X$ is extremely proximal. □

Lemma 3.13. If $\mathcal{K} \subseteq \mathscr{C}(X)$ is a nonempty convex subset of $\mathscr{C}(X)$, such that $\mathcal{K} \setminus \{X\} = \mathscr{C}(X) \setminus \{X\}$, then $\mathcal{K} = \mathscr{C}(X)$.

(Proof) Assume that $\mathcal{K} \subseteq \mathscr{C}(X)$ is a convex subset of $\mathscr{C}(X)$ such that
$\mathcal{K} \setminus \{X\} = \mathscr{C}(X) \setminus \{X\}$. It suffices to show that $X \in \mathcal{K}$.
If X is singleton, then $\mathcal{C}(X) = \{X\}$ and since $\mathcal{K}$ is non empty, we must have $\mathcal{K} = \{X\} = \mathscr{C}(X)$.
Assume X contains two distinct points x, y. Let U, V be disjoint open subsets of X such that $x \in U$ and $y \in V$. Then $y \notin \overline{U}$, so $\overline{U} \in \mathscr{C}(X) \setminus \{X\}$. Thus, $\overline{U} \in \mathcal{K}$. On the other hand, since $x \in U, X \setminus U \in \mathscr{C}(X) \setminus \{X\}$, so that $X \setminus U \in \mathcal{K}$. Thus, $X = \overline{U} \cup (X \setminus U) \in \mathcal{K}$, since $\mathcal{K}$ is a convex subset of $\mathscr{C}(X)$. □

Our main result gives parallel characteristics of proximal and extremely proximal for minimal actions.

Theorem 3.14. Suppose $G \curvearrowright^{\alpha} X$ is minimal.

1. $G \curvearrowright^{\alpha} X$ is proximal if and only if $\mathscr{F}(X)$ has no nonempty proper G-invariant, convex, relatively closed subsets.

2. $G \curvearrowright^{\alpha} X$ is extremely proximal if and only if $\mathscr{C}(X) \setminus \{x\}$ has no nonempty proper G-invariant, relatively convex, relatively closed subsets.

(Proof) 1. Suppose $G \curvearrowright^{\alpha} X$ is proximal. Let $K \subseteq \mathscr{F}(X)$ be nonempty G-invariant, convex, relatively closed. By Lemma 3.11 it is enough to show that K contains a singleton subset of X.

Let $A \in K$. Now $A = \{x_1, \ldots, x_k\}$. Since $G \curvearrowright^{\alpha} X$ is proximal, then there is $(g_i^1)_{i \in I} \in G$ such that $\lim g_i^1.x_1 = \lim g_i^1.x_2 = x_1^1$. Since X is compact, then there is a subnet of $(g_i^1.x_3)_{i\in I}$ say $(g_i^3.x_3)_{i\in I}$ (after relabeling) such that $(g_i^3.x_3)_{i\in I} \to x_1^3$.

Using the compactness of X, we get a subnet of $(g_i^3.x_4)_{i\in I}$ say $(g_i^4.x_4)_{i\in I}$ (after relabeling) such that $(g_i^4.x_4)_{i\in I} \to x_1^4$.

Proceeding this way, we get a subnet of $(g_i^{k-1}.x_k)_{i\in I}$ say $(g_i^k.x_k)_{i\in I}$ (after relabeling) such that $(g_i^k.x_k)_{i\in I} \to x_1^k$. Note that $(g_i^k)_{i\in I}$ is a subnet of $(g_i^1)_{i\in I}$. By Lemma 3.10, we get $(g_i^k).\{x_1, \ldots, x_k\} = \{g_i^k.x_1, \ldots, g_i^k.x_k\} \to \{x_1^1, \ldots, x_1^r\}$, where $r \leq k-1$. Since K is G-invariant and relatively closed, then $\{x_1^1, \ldots, x_1^r\} \in K$.

Repeating the same argument on $\{x_1^1, \ldots, x_1^r\}$, we get a sequence $(g_i^{k+1})_{i\in I}$ such that $g_i^{k+1}.\{x_1^1, \ldots, x_1^r\} \to \{x_2^1, \ldots, x_2^s\}$, where $s \leq r-1 < k$ and $\{x_2^1, \ldots, x_2^s\} \in K$. Proceeding this way, we get $(g_i^p)_{i\in I} \in G$ such that $g_i^p.\{x_p^1, x_p^2\} \to \{x\}$, where $\{x_p^1, x_p^2\} \in K$. and $\{x\} \in K$.

Conversely, let $x, y \in X$. Consider $F = G.\{x,y\} \subseteq \mathscr{C}(X)$. Since $F \subseteq \mathscr{C}(X)$ is G-invariant, then $\overline{co}(F) \cap \mathscr{F}(X) \subseteq \mathscr{F}(X)$ is G-invariant, convex and relatively closed in $\mathscr{F}(X)$. Hence $\overline{co}(F) \cap \mathscr{F}(X) = \mathscr{F}(X)$.

Density of $\mathscr{F}(X)$ in $\mathscr{C}(X)$ gives $\overline{co}(F) = \mathscr{C}(X)$. By the Milman theorem in the context of $\mathscr{C}(X)$, we get $\{\{x\}; x \in X\} \subseteq \overline{F} = \overline{G.\{x,y\}}$. i.e. there is a net $(g_i)_{i\in I}$ such that

$g_i.\{x,y\} \to \{z\}$ for some $z \in X$ i.e. $g_i.x \to z$ and $g_i.y \to z$.

2. Suppose $G \curvearrowright^{\alpha} X$ is extremely proximal. Let $\mathcal{K} \subseteq \mathscr{C}(X) \setminus \{X\}$ be a nonempty proper, G-invariant, relatively convex and relatively closed set. So there exists a closed set L in $\mathscr{C}(X)$ such that $\mathcal{K} = (\mathscr{C}(X) \setminus \{X\}) \cap L$. By Lemma 3.12, for any $C \in \mathcal{K}$, we have $\{\{x\}; x \in X\} \subseteq \overline{G.C} \subseteq \mathcal{K} \subseteq L$.

   So $co\{\{x\}; x \in X\} \subseteq \mathcal{K} \subseteq L$.

   Since $co\{\{x\}; x \in X\}$ is dense in $\mathscr{C}(X)$, then $\mathscr{C}(X) = \overline{co}\{\{x\}; x \in X\} \subseteq L$. Therefore $L = \mathscr{C}(X)$. Hence $\mathcal{K} = \mathscr{C}(X) \setminus \{X\}$.

   Conversely, suppose that there is no nonempty proper, G-invariant, relatively convex and relatively closed subset of $\mathscr{C}(X) \setminus \{X\}$. Let $C \in \mathscr{C}(X) \setminus \{X\}$, then $\overline{co}(G.C)$ is closed and convex subset of $\mathscr{C}(X)$. Therefore $\overline{co}(G.C) \setminus \{X\} = \overline{co}(G.C) \cap (\mathscr{C}(X) \setminus \{X\}$, where $\overline{co}(G.C) \setminus \{X\}$ is nonempty G-invariant, relatively closed and relatively convex subset of $\mathscr{C}(X) \setminus \{X\}$. Thus by hypothesis, $\overline{co}(G.C) \setminus \{X\} = \mathscr{C}(X) \setminus \{X\}$. By lemma we get that $\overline{co}(G.C) = \mathscr{C}(X)$. Using Milman's theorem for G.C, we get that $\{\{x\}; x \in X\} \subseteq \overline{G.C}$. By lemma 4, we get that $G \curvearrowright^{\alpha} X$ is extremely proximal.

□

Question 3.15. Is there any collection $\mathscr{A}(X)$, where $\mathscr{F}(X) \subseteq \mathscr{A}(X) \subseteq \mathscr{C}(X)$ such that $G \curvearrowright^{\alpha} X$ is strongly proximal if and only if $\mathscr{A}(X)$ has no nonempty proper G-invariant, convex, relatively closed subsets?

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