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Equivalent Constructions of Cartan Pairs

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EQUIVALENT CONSTRUCTIONS OF CARTAN PAIRS

by

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Abstract

EQUIVALENT CONSTRUCTIONS OF CARTAN PAIRS

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Feldman and Moore [4] introduce Cartan subalgebra of the von Neumann algebra M on a separable Hilbert space $\mathcal{H}$ from the natural subalgebra of $M(R, \sigma)$, the twisted algebra of matrices over the relation R on a Borel space $(X, \mathcal{B}, \mu)$. They show that if M has a Cartan subalgebra A, then $M \cong M(R, \sigma)$ where A is the twisted algebra onto the diagonal subalgebra $L^{\infty}(X, \mu)$. The relation R is unique to isomorphism and the orbit of the two-cohomology class on R in the torus $\mathbf{T}$, which is the automorphism group of R, is also unique. Three decades later, based on Feldman-Moore work and utilizing étale groupoids from $C_r^*(G, \Sigma)$, Renault [9] constructs equivalent Cartan pairs. Nearly another decade later, using extensions of inverse semigroups from extensions of Cartan inverse monoids and Feldman-Moore work, Donsig, Fuller, and Pitts [2] construct other equivalent Cartan pairs. In this paper, we study all Cartan pairs of Feldman and Moore, Renault, and Donsig et al. Our objective is to show that these Cartan pairs are equivalent.

Chapter 1

Introduction

First we would like to provide a brief review of von Neumann algebras (known as W^*-algebras), the Gelfand-Naimark-Segal (GNS) construction and representation, Cartan MASA, and Cartan subalgebra. Notice that we use Dixmier [1] for many basic concepts and theorems of von Neumann algebras.

Definition 1.1. Let $\mathcal{H}$ denote a Hilbert space. A sequence $\{T_i\}$ *converges weakly* (or converges in the weak operator topology) to T in $\mathcal{H}$ if $\langle T_i x, y\rangle \to \langle Tx, y\rangle$ for all $y \in \mathcal{H}$. A sequence $\{T_i\}$ *converges strongly* (or converges in the strong operator topology) to T in $\mathcal{H}$ if $T_i x \to Tx$ for all $x \in \mathcal{H}$.

Definition 1.2. A *von Neumann algebra* M on $\mathcal{H}$ is a $*$-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$ that is weakly closed. Then M is a norm closed. This implies that M is a C^*-algebra.

Exposition 1.3 ($*$-algebra)**.** A $*$-*algebra* A is a $*$-ring with involution which is an associative algebra over a commutative $*$-ring with involution such that $(ra)^* = r'a^*$ for all $a \in A$ and $r \in R$.

Exposition 1.4 (Norm)**.** Let V be a $\mathbf{R}$-vector space. A *norm* on V is a function $\|\cdot\| : V \to \mathbf{R}$ satisfying the following conditions.

1. (Positive definite) $\|x\| \geq 0$ for all $x \in V$, and $\|x\| = 0$ iff $x = 0$.

2. (Triangle inequality) $\| x + y \| \leq \|x\| + \|y\|$ for all $x, y \in V$.

3. (Homogeneity) $\| \alpha x \| = |\alpha| \|x\|$ for all $\alpha \in \mathbf{R}$ and $x \in V$.

Exposition 1.5 (C^*-algebra)**.** A C^*-*algebra* is a Banach algebra B over C with the following additional properties.

1. There is a conjugate linear map $*\colon B \to B$ satisfying

   $(b^*)^* = b$ (an involution) and $(b_1 b_2)^* = (b_2)^*(b_1)^*$ for all $b_1, b_2 \in B$.

2. The C^*-identity holds or $\| b^* b \| = \|b\|^2$ for all $b \in B$.

Definition 1.6 (Commutant and Double Commutant)**.** Let $S \subseteq \mathcal{B}(\mathcal{H})$. The *commutant* of S is

$$S' = \{T \in \mathcal{B}(\mathcal{H}) : \forall s \in S, Ts = sT\}.$$

Denote S'' as the *double commutant* so $S'' = (S')'$.

Remark 1.7. Note that $S \subseteq S''$. If S is self-adjoint, then S' and S'' are von Neumann algebras. Recall that a bounded linear operator L: $\mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is self-adjoint if $L^* = L$ where the adjoint $L^* \in \mathcal{B}(\mathcal{H})$ of an operator $L \in \mathcal{B}(\mathcal{H})$ is defined by $\langle x, Ly \rangle = \langle L^* x, y \rangle$ for all $x, y \in \mathcal{H}$. A set $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is self-adjoint if every operator in $\mathcal{S}$ is self-adjoint. By the following theorem, if the identity operator $I_{\mathcal{H}} \in S$, then S'' is the smallest von Neumann algebra containing S.

Exposition 1.8 (Bounded Linear Operator)**.** Let V and W be vector spaces over a field F. A map $T : V \to W$ is *linear* if $T(x + y) = T(x) + T(y)$ and $T(\alpha x) = \alpha T(x)$ for all $x, y \in V$ and all $\alpha \in F$. A linear map $T : V \to W$ is a *linear operator.*

Let V and W be normed vector spaces. An operator $T : V \to W$ is *bounded* if there exists a constant $K \geq 0$ such that $\| Tx \| \leq K\|x\|$ for all $x \in V$.

Theorem 1.9 (von Neumann Bicommutant Theorem)**.** Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra with $I_{\mathcal{H}} \in M$. The following are equivalent:

1. M is WOT-closed.

2. M is SOT-closed.

3. $M = M''$.

If M is SOT-closed, then M is unital. Note that the unit of M and the identity operator in $\mathcal{B}(\mathcal{H})$ may not be the same, but we can always cut down by a projection. Let $\mathcal{I} \in M$ be the identity element, then we can replace $\mathcal{H}$ by $\mathcal{I}\mathcal{H}$ and M by $\mathcal{I}M\mathcal{I}$.

Theorem 1.10 (Abelian von Neumann Algebras, Dixmier [1]). If M is a commutative von Neumann algebra on $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is assumed to be separable, then there exists a second countable compact Hausdorff space X and a Borel measure μ on X such that $M \cong L^\infty(X, \mu)$.

Exposition 1.11 (Borel Measure). Let X be a locally compact Hausdorff space and let $\mathcal{B}(X)$ be the smallest σ-algebra which contains open sets of X. A *Borel measure μ on X* is any measure μ defined on the σ-algebra of Borel sets.

Exposition 1.12 ($L^\infty(X, \mu)$). $L^\infty(X, \mu)$ is the set of all (equivalence classes of μ-a.e. equal) measurable functions having a finite essential supremum:

$$L^\infty(X, \mu) = \{f : X \to C \mid esssup_{x \in X} \mid f(x) \mid < \infty\}$$

.

Definition 1.13 (*Gelfand-Naimark-Segal (GNS) Construction*). Let $S(M)$ be the state space of a von Neumann algebra M. Recall that a state of M is a positive linear functional on M of norm one. Each $\varphi \in S(M)$ gives us a representation of M. Define a pre-inner product on M by $\langle a, b \rangle = \varphi(b^* a)$. Let

$$N_\varphi = \{a \in M : \varphi(a^* a) = 0\}.$$

Notice that φ induces an inner product on M/N_φ. Then let $\mathcal{H}_\varphi$ be the Hibert space completion of M/N_φ. Define the map $\pi_\varphi : M \to \mathcal{B}(\mathcal{H}_\varphi)$ by $\pi_\varphi(a)(b + N_\varphi) = ab + N_\varphi$.

We refer this as the *GNS construction* and π_φ as the *GNS representation.* Notice that if

$\mathcal{H} = \bigoplus_{\varphi \in S(M)} \mathcal{H}_\varphi$ and $\pi = \bigoplus_{\varphi \in S(M)} \pi_\varphi$, then π is a universal representation of M which is faithful and non-degenerate representation of M on $\mathcal{H}$.

Exposition 1.14 (Pre-inner product). Let X be a vector space over a field F. A *pre-inner product* is a positive Hermitian sesquilinear form $(\cdot \mid \cdot)$.

Exposition 1.15 (Faithful Representation). A state φ on a C^*-algebra A is faithful if

$\varphi(a^*a) = 0$ iff $a = 0$ for $a \in A$. The GNS representation, π_φ, of the faithful state φ is *faithful*.

Exposition 1.16 (Nondegenerate Representation)**.** Let A be a C^*-algebra and $\mathcal{H}$ be a Hilbert space. Define a *non-degenerate representation* as $\{\pi(a)v : a \in A, v \in \mathcal{H}\}$ which is dense in $\mathcal{H}$. By Zorn's lemma, a non-degenerate representation is equivalent to an orthogonal direct sum of cyclic representations.

Definition 1.17 (Normal Expectation)**.** Let A be a von Neumann algebra. Let B be a von Neumann subalgebra of A. Define an *expectation* from A on B as a positive map $\varphi : A \to B$ which preserves identity and for all $S \in B$ and for all $Y \in A$, $\varphi(SY) = S\varphi(Y)$. Let $\{Y_\beta\}$ be a set of uniformly bounded self-adjoint operators of A. If $\varphi(\sup Y_\beta) = \sup \varphi(Y_\beta)$, then φ is a normal expectation.

Definition 1.18 (Conditional Expectation)**.** A *conditional expectation* is a completely positive contraction $E : A \to B$ such that for all $b \in B$, $E(b) = b$ and for all $x \in A$, $E(bx) = bE(x)$ and $E(xb) = E(x)b$. By Tomiyama [10], E is a conditional expectation iff $E : A \to A$ is an idempotent with norm one.

Definition 1.19 (Cartan MASA)**.** Let M be a von Neumann algebra. Define a maximal abelian subalgebra (MASA) D in M as a *Cartan MASA* if

1. There exists a normal, faithful conditional expectation E from M onto D.

2. The set of groupoid normalizers $N(M, D)$ spans a weak-$*$dense subset of M. That is, the set of unitaries $U \in M$ such that $UDU^* = U^*DU = D$ spans a weak-$*$dense subset in M. If we use partial isometries $V \in M$ such that $VDV^*, V^*DV \subseteq D$ spans a weak-$*$ dense subset in M.

If D is a Cartan MASA in M, the pair (M, D) is a *Cartan pair*.

The definition of a Cartan pair from von Neumann algebra perspective is slightly different from C^*-algebra perspective. Below we describe how a Cartan pair is defined for C^*-algebras.

Definition 1.20 (Groupoid)**.** A *groupoid* is a set $\mathcal{G}$ together with a distinguished $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a multiplication map $(\alpha, \beta) \to \alpha\beta$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$ and an inverser map $\gamma \to \gamma^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ such that

1. For all $\gamma \in \mathcal{G}$, $(\gamma^{-1})^{-1} = \gamma$;

2. If (α, β) and $(\beta, \gamma) \in \mathcal{G}^{(2)}$, then $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma) \in \mathcal{G}^{(2)}$, and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$; and

3. $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$, and $\forall(\gamma, \eta) \in \mathcal{G}^{(2)}$, we have $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$.

Definition 1.21 (Cartan Subalgebra from C^*-Algebras)**.** Let A be a C^*-algebra and B be a C^*-subalgebra of A. Then $B \subseteq A$ is a Cartan subalgebra if

1. B is a MASA.

2. B contains an approximate unit of A.

3. Let the set of normalizers of B, $N(B)$, generate A as a C^*-algebra where

$$N(B) = \{a \in A : aBa^* \subseteq B \text{ and } a^*Ba \subseteq B\}.$$

   Then B is *regular* in A.

4. There exists a faithful conditional expectation $E : A \twoheadrightarrow B$.

Then (A, B) is a Cartan pair.

Renault [9] proves that in the reduced C^*-algebra of a topologically principal Hausdorff étale groupoid with a twist, the subalgebra, $C_0(G^{(0)})$, which corresponds to the unit space $G^{(0)}$, is a Cartan subalgebra. Conversely, every Cartan pair arises from a twisted groupoid. Furthermore, Renault shows that the Cartan pair completely determines the groupoid and vice versa.

Instead of utilizing a 2-cocycle on an equivalence relation R as in the Feldman- Moore work, Donsig et al. [2] use an extension of the Cartan inverse monoid by the abelian inverse semigroup of partial isometries in the C^*-algebra generated by its idempotents. Donsig et al. [2] consider their work conceptually simpler compared to Feldman-Moore work. The authors build a Cartan MASA in the extension's von Neumann algebra.

We consider our work as a contribution to the future study of von Neumann algebras, and C*-algebras, Cartan subalgebras, especially on the equivalent constructions between étale groupoids and extensions (exact sequence) of inverse semigroups. We organize our paper in the following order: assuming that our readers are familiar with Feldmann-Moore work, throughout Chapter 2, we provide necessary background of Renault [9] and Donsig et al. [2]'s work. During Chapter 3, we prove the equivalence of the constructions of Feldman-Moore and Renault Cartan pairs. Note that if the authors provide the proof(s) for one direction or both, we summarize their proofs. In Chapter 4, we analyze proofs from Donsig et al. to show that Feldman-Moore and Donsig et al. Cartan pairs are equivalent. During Chapter 5, we leave our readers the main message of our work and possible extensions.

Chapter 2

Preliminaries

In this section we would like to provide necessary background of how Donsig et al. [2] and Renault [9] construct their Cartan pairs which include important topics such as Munn congruence, Cartan inverse monoids, extensions of inverse semigroups, topologically principal, étale groupoid, and so on.

2.1 Necessary Background of Donsig et al. Cartan Pair

Definition 2.1 (Inverse Semigroups)**.** A semigroup $\mathcal{S}$ is an inverse semigroup if for all $s \in \mathcal{S}$, there exists a unique inverse element s^* such that $ss^*s = s$ and $s^*ss^* = s^*$. Denote $\mathcal{E}(\mathcal{S})$ as the idempotents in an inverse semigroup $\mathcal{S}$. Note that idempotents form an abelian semigroup. For each $s \in \mathcal{S}$, we have $ss^* \in \mathcal{E}(\mathcal{S})$. An inverse semigroup $\mathcal{S}$ has a natural partial order $s \leqslant t$ iff $s = te$ for some idempotent $e \in \mathcal{E}(\mathcal{S})$.

Definition 2.2 (Extensions of Inverse Semigroups)**.** Suppose $\mathcal{S}$ and $\mathcal{P}$ are inverse semigroups. Let $\pi : \mathcal{P} \to \mathcal{S}$ be a surjective homomorphism such that $\pi|_{\mathcal{E}(\mathcal{P})}$ is an isomorphism from $\mathcal{E}(\mathcal{P})$ to $\mathcal{E}(\mathcal{S})$. Define an idempotent separating extension of $\mathcal{S}$ by $\mathcal{P}$ as an inverse semigroup $\mathcal{G}$ with $\mathcal{P} \overset{i}{\hookrightarrow} \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$. Note that i denotes an injective homomorphism; q represents a surjective homomorphism, and $\pi = q \circ i$. Also note that $\mathcal{E}(\mathcal{P}) \cong \mathcal{E}(\mathcal{G}) \cong \mathcal{E}(\mathcal{S})$ and $q(g) \in \mathcal{E}(\mathcal{S})$ iff $g = i(p)$ for $p \in \mathcal{P}$.

Definition 2.3 (Fundamental and Clifford)**.** An inverse semigroup $\mathcal{S}$ is *fundamental* if for $s_1, s_2 \in \mathcal{S}$, $s_1 e s_1^* = s_2 e s_2^*$ for all $e \in \mathcal{E}(\mathcal{S})$ only when $s_1 = s_2$.

An inverse semigroup is *Clifford* if $s^*s = ss^*$ for all $s \in \mathcal{S}$.

Definition 2.4 (Congruence)**.** Let $\mathcal{G}$ be an inverse semigroup. An equivalence relation R on $\mathcal{G}$ is a *congruence* if $(v_1, v_2), (w_1, w_2) \in R$ implies $(v_1w_1, v_2w_2) \in R$.

Definition 2.5 (Munn Congruence)**.** Let $\mathcal{G}$ be an inverse semigroup and R be a congruence. The quotient of $\mathcal{G}$ by R gives an inverse semigroup $\mathcal{S}$. Let $q : \mathcal{G} \to \mathcal{S}$ denote the quotient map and $\mathcal{P} = \{v \in \mathcal{G} : q(v) \in \mathcal{E}(S)\}$. Then $\mathcal{P}$ is an inverse semigroup and $\mathcal{G}$ is an extension of $\mathcal{S}$ by $\mathcal{P}$.

Define the *Munn congruence* R_M by s is related to t iff $ses^* = tet^*$ for all $e \in \mathcal{E}(\mathcal{S})$ and for all $s, t \in \mathcal{S}$. Note that R_M is the maximal idempotent separating congruence on $\mathcal{G}$ and an inverse semigroup is *fundamental* if its Munn congruence is the equality relation where

$$R_M = \{(v_1, v_2) \in \mathcal{G} \times \mathcal{G} : \forall e \in \mathcal{E}(\mathcal{G}), v_1 e v_1^* = v_2 e v_2^*\}.$$

Let $\mathcal{G}$ be an idempotent separating extension of $\mathcal{S}$ by $\mathcal{P}$ as above. Then $\mathcal{P}$ is a Clifford inverse semigroup.

Definition 2.6 (Groupoid Normalizer)**.** Donsig et al. [2] later use the Munn congruence on a groupoid normalizer. Let us first define *the normalizer* of $\mathcal{D}$ as

$$\mathcal{N}(\mathcal{D}) = \{v \in \mathcal{M} \text{ a partial isometry: } v^*\mathcal{D}v \subseteq \mathcal{D} \text{ and } v\mathcal{D}v^* \subseteq \mathcal{D}\}.$$

Then denote $\mathcal{GN}(\mathcal{M}, \mathcal{D})$ as the collection of all *groupoid normalizers* of a MASA $\mathcal{D}$ in a von Neumann algebra M. Since $\mathcal{GN}(\mathcal{M}, \mathcal{D})$ is an inverse semigroup, define the *Munn congruence* R_M on $\mathcal{GN}$ as

$$R_M = \{(v_1, v_2) \in \mathcal{GN} \times \mathcal{GN} : \forall e \in \mathcal{E}(\mathcal{GN}), v_1 e v_1^* = v_2 e v_2^*\}.$$

Thus R_M, in this case, is the maximal idempotent separating congruence on $\mathcal{GN}$ and the quotient of $\mathcal{GN}$ by R_M is a fundamental inverse semigroup $\mathcal{S}$.

Definition 2.7. Donsig, Fuller, and Pitts [2] define a *Cartan inverse monoid* as an inverse semigroup $\mathcal{S}$ which satisfies the properties below:

1. $\mathcal{S}$ is a meet lattice under the natural partial order on $\mathcal{S}$.

2. $\mathcal{S}$ contains 0 and 1.

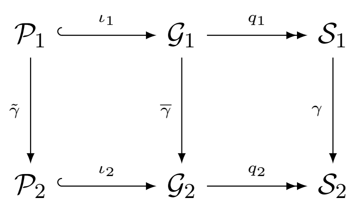

Figure 2.1: Donsig et al. [2]'s Diagram Commutes.

3. $\mathcal{E}(\mathcal{S})$ is maximal abelian in $\mathcal{S}$.

4. $\mathcal{E}(\mathcal{S})$ is a hyperstonean boolean algebra. That is, the idempotents are the projection lattice of an abelian von Neumann algebra.

5. For every pairwise orthogonal family $F \subseteq \mathcal{S}$, $\bigvee F$ exists in $\mathcal{S}$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be isomorphic Cartan inverse monoids from a von Neumann algebra. Let $\mathcal{P}_j$ be the partial isometries in $\mathcal{D}_j$ for $j = 1, 2$. Extensions $\mathcal{G}_j$ of $\mathcal{S}_j$ by $\mathcal{P}_j$ are equivalent if there is an isomorphism $\underline{\gamma} : \mathcal{G}_1 \to \mathcal{G}_2$ such that the diagram in figure 2.1 commutes. A surjective homomorphism $q : \mathcal{G} \to \mathcal{S}$, is an idempotent which separates the extension of $\mathcal{S}$ by $\mathcal{P}$ if an embedding ι of $\mathcal{P}$ into $\mathcal{G}$ satisfies the following:

1. For some $p \in \mathcal{P}$, $q(g) \in \mathcal{E}(\mathcal{S})$ iff $g = \iota(p)$

2. $q \circ \iota = \pi$.

To construct a Cartan pair from an extension of a Cartan inverse monoid, Donsig et al. [2] build a corresponding representation for extensions of Boolean inverse monoids. Note that $\mathcal{D}$ is a MASA of the von Neumann algebra $\mathcal{M}$. Given $\mathcal{P} \to \mathcal{G} \overset{q}{\to} \mathcal{S}$, an extension of a Boolean inverse monoid $\mathcal{S}$, the authors represent $\mathcal{G}$ by partial isometries which acts on a Hilbert space by several important steps. One of these is to construct a $\mathcal{D}$-valued reproducing kernel and a right Hilbert $\mathcal{D}$-module to use the interior tensor product $\mathcal{U} \otimes_\pi \mathcal{H}$ in order to obtain a class of representations of $\mathcal{G}$ on $\mathcal{H}$.

Definition 2.8 ($\mathcal{D}$-valued Reproducing Kernel Hilbert Space $\mathcal{D}$-module)**.** Let $f : \mathcal{S} \to \mathcal{G}$ such that $f \circ q = id$. Donsig et al. [2] show that the order-preserving section, $f(s) \leq f(t)$ when $s \leq t$ exists and $f : \mathcal{E}(\mathcal{S}) \to \mathcal{E}(\mathcal{G})$ is an isomorphism. Define $K : \mathcal{S} \times \mathcal{S} \to \mathcal{D}$ by $K(s,t) = f(s^*t \wedge 1)$.

The idempotent $s^*t \wedge 1$ is the minimal idempotent e such that $se = te = s \wedge t$. Thus $K(s,t)$ is the idempotent in $\mathcal{G}$ which defines $f(s) \wedge f(t)$. For $c_1, \cdots, c_k \in \mathbf{C}$ and $s_1, \cdots, s_k \in S$, the map K is positive. In notation, $\sum_{i,j} \bar{c}_i c_j K(s_i, s_j) \geq 0$.

For each $s \in \mathcal{S}$, Donsig et al. [2] define a kernel $k_s : S \to \mathcal{D}$ by $k_s(t) = K(t,s)$. Let $U_0 = span\{k_s : s \in \mathcal{S}\}$. Note that $K > 0$ implies that

$$\left\langle \sum c_i k_{s_i}, \sum d_j k_{t_j} \right\rangle = \sum_{i,j} \bar{c}_i d_j K(s_i, t_j)$$

known as the $\mathcal{D}$-valued inner product on U_0 where U is a completion of U_0. Thus U is a reproducing kernel Hilbert $\mathcal{D}$-module of functions from $\mathcal{S}$ onto $\mathcal{D}$.

Definition 2.9 (Left Representation of $\mathcal{G}$)**.** Choose $g \in \mathcal{G}$ and let $\lambda(g)$ be an adjointable operator on U such that $\lambda(g)k_s = k_{q(g)s}\sigma(g.s)$ where $\sigma : \mathcal{G} \times \mathcal{S} \to \mathcal{P}$ is a cocycle-like function determined by $gf(s) = f(q(g)s)\sigma(g,s)$.

We can factor elements of the form $gf(s)$ into the product of an element in $f(\mathcal{S})$ by an element in $\mathcal{P}$. We call $\lambda : \mathcal{G} \to L(U)$ the left representation of $\mathcal{G}$ by partial isometries.

Let π be a faithful representation of $\mathcal{D}$ on a Hilbert space $\mathcal{H}$. Obtain a Hilbert space $U \otimes_\pi \mathcal{H}$ by completing $U \otimes \mathcal{H}$ with respect to the inner product $\langle u_1 \otimes h_1, u_2 \otimes h_2 \rangle = \langle h_1, \pi(\langle u_1, u_2 \rangle) h_2 \rangle$. Therefore π determines a faithful representation $\hat{\pi}$ of $L(U)$ on the Hilbert space $U \otimes_\pi \mathcal{H}$ by
$\hat{\pi}(T)(a \otimes h) = (Ta) \otimes h$.

Define a faithful representation of $\mathcal{G}$ on the Hilbert space $U \otimes_\pi \mathcal{H}$ by $\lambda_\pi : g \to \hat{\pi}(\lambda(g))$.

Definition 2.10 (Cartan Pair)**.** Let $\mathcal{M}_q = \lambda(\mathcal{G})''$ and $\mathcal{D}_q = \lambda(\mathcal{E}(\mathcal{S}))''$. Donsig et al. [2] define $(\mathcal{M}_q, \mathcal{D}_q)$ as a Cartan pair with the following properties:

1. $\mathcal{D}_q \cong C(\widehat{\mathcal{E}(\mathcal{S})})$.

2. The pair $(\mathcal{M}_q, \mathcal{D}_q)$ is independent of π and j.

3. The conditional expectation $\mathcal{M}_q \to \mathcal{D}_q$ is induced from the map $\mathcal{S} \to \mathcal{E}(\mathcal{S})$. That is, $s \mapsto s \wedge 1$.

4. The extension associated to $(\mathcal{M}_q, \mathcal{D}_q)$ is equivalent to $\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$.

Theorem 2.11 (Donsig et al. [2])**.** The extension $\mathcal{P} \hookrightarrow \mathcal{G} \stackrel{q}{\to} \mathcal{S}$ determines a Cartan pair $(\mathcal{M}, \mathcal{D})$ which is unique up to isomorphism if $\mathcal{S}$ is a Cartan inverse monoid and the extension is given by $\mathcal{P} = C(\widehat{\mathcal{E}(\mathcal{S})})$. Moreover, equivalent extensions determine isomorphic Cartan pairs.

Every Cartan pair $(\mathcal{M}, \mathcal{D})$ determines uniquely an extension of a Cartan inverse semigroup $\mathcal{S}$ by $\mathcal{P}$ with $\mathcal{P} \hookrightarrow \mathcal{G} \stackrel{q}{\to} \mathcal{S}$.

2.2 Necessary Background of Renault's Cartan Pair

For the rest of this chapter, we want to review important definitions of Renault [9]'s paper.

Definition 2.12 (Topologically Principal Groupoids)**.** Let G be a groupoid defined in chapter 1. Denote $G^{(0)}$ as the unit space. Let r be the range and s be the source $G \to G^{(0)}$. The fiber of the range are $G^x = r^{-1}(x)$ and the source maps are $G_y = s^{-1}(y)$. Define the *isotropy bundle* as $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$.

Suppose that the groupoid G is a topological space and the structure maps are continuous where $G^{(2)}$, the set of composable pairs, has the topology induced by $G \times G$ and $G^{(0)}$ has the topology induced by G.

By Renault [9], a topological groupoid G is *étale* when its range and source maps are local homeomorphisms $G \to G^{(0)}$. Then the range and source maps are open.

An étale groupoid G is *principal* if the isotropy bundle $G' = G^{(0)}$ the unit space. An étale groupoid G is *topologically principal* if the set of points of the unit space $G^{(0)}$ with trivial isotropy is dense in G.

Definition 2.13 (Pseudogroup and Groupoid of Germs)**.** Let X be a topological space. A *pseudogroup* on X is a family $\mathcal{G}$ of partial homeomorphisms of X stable under composition and inverse. Given a pseudogroup $\mathcal{G}$ on X, define the *groupoid of germs* as

$$G = \{[x, \varphi, y], \varphi \in \mathcal{G}, y \in dom(\varphi), x = \varphi(y)\}$$

where $[x, \varphi, y] = [x, \psi, y]$ iff there exists a neighborhood U of y in X such that $\varphi|_U = \psi|_U$.

Let $X = G^{(0)}$. An étale groupoid G defines a pseudogroup $\mathcal{G}$ on X in the sense that G has a cover of open bisections. Rcall that a bisection is a subset $S \subset G$ which is both an r-section and an s-section. (A subset of a groupoid is an r-section or an s-section if the restriction of r or s to the subset is injective.) The open bisections of an étale groupoid G form an inverse semigroup $S(G)$ with $ST = \{\gamma\gamma' : (\gamma, \gamma') \in (S \times T) \bigcap G^{(2)}\}$ and the image of S by the inverse map is S^{-1}.

Definition 2.14 (Effective Groupoid)**.** Let G be an étale groupoid over X and let S be the inverse semigroup of its open bisections. Then G is an *effective groupoid* if the interior of G' is reduced to $G^{(0)}$.

Proposition 2.15 (Topologically Principal and Effective Groupoid, Renault [9], 3.6)**.** Let G be an étale groupoid.

1. If G is Hausdorff and topologically principal, then it is effective.
2. If G is a second countable effective groupoid and its unit space $G^{(0)}$ has the Baire property, then it is topologically principal.

Definition 2.16 (Twisted Groupoid C^*-algebra)**.** Let G and Σ be topological groupoids. Given $\mathbf{T}$ is the circle group. Define a twisted groupoid as a central groupoid extension $\mathbf{T} \times G^{(0)} \hookrightarrow \Sigma \twoheadrightarrow G$. Note by assumption, Σ is a principal $\mathbf{T}$- space and $\Sigma/\mathbf{T} = G$.

Let (G, λ) be a Hausdorff locally compact second countable groupoid with Haar system and suppose Σ is a twist over G. Denote the space of continuous sections with compact support of the line bundle associated with Σ as $C_c(G, \Sigma)$. Define a $*$-algebra by the operations:

$$f * g(\sigma) = \int f(\sigma\tau^{-1})g(\tau)d\lambda_{s(\sigma)}(\dot{\tau})$$

where $\dot{\tau} \in G$, the image of $\tau \in \Sigma$, and $f^*(\sigma) = \overline{f(\sigma^{-1})}$.

For $x \in G^{(0)}$, define the Hilbert space $\mathcal{H}_x = L^2(G_x, L_x, \lambda_x)$ of square-integrable sections of the line bundle $L_x = L_{|G_x}$. For $f \in C_c(G, \Sigma)$, define the operator $\pi_x(f)$ on $\mathcal{H}_x$ as

$$\pi_x(f)\mathcal{E}(\sigma) = \int f(\sigma\tau^{-1})\mathcal{E}(\tau)d\lambda_x(\dot{\tau})$$

The space of sections $C_0(G^{(0)}, \mathcal{H})$ is the right C^*-module over $C_0(G^{(0)})$ and π is a representation of $C_c(G, \Sigma)$ on this C^*-module. The *reduced C^*-algebra*, denoted as $C_r^*(G, \Sigma)$, is the completion of $C_c(G, \Sigma)$ with respect to the norm $\|f\| = \sup_x \|\pi_x(f)\|$.

Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C_r^*(G, \Sigma)$ and $B = C_0(G^{(0)})$. Choose $a \in A$. By Renault [9], a commutes with every element of B iff its open support, $supp'(a) \subseteq G'$. Note that B is a MASA iff G is topologically principal.

With the same assumption that (G, Σ) is a twisted étale Hausdorff locally compact second countable groupoid, let $P : C_r^*(G, \Sigma) \to C_0(G^{(0)})$ be the restriction map obtained by restricting elements of the dense subalgebra $C_c(G, \Sigma)$ and extend the map by the continuity property to the entire C^*-algebra. Then P is a faithful conditional expectation onto $C_0(G^{(0)})$. Moreover, if G is topologically principal, then P is the unique conditional expectation onto $C_0(G^{(0)})$.

Exposition 2.17 (Haar Measure)**.** If a groupoid G is locally compact, a left (right) *Haar measure* on G is a nonzero left-invariant (right-invariant) Radon measure μ on G. Recall that a Radon measure on a topological space is a Borel measure which is finite on compact sets, inner regular on open sets, and outer regular on Borel sets.

Definition 2.18. Let B be a sub C^*-algebra of a C^*-algebra A. Then B is *regular* if its normalizer

$N(B) = \{n \in A : nBn^* \subset B \text{ and } n^*Bn \subset B\}$ generates A as a C^*-algebra.

Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid and $A = C_r^*(G, \Sigma)$. Then $B = C_0(G^{(0)})$ is regular sub C^*-algebra of A. Furthermore, if (G, Σ) is also a topologically principal groupoid, then B is a Cartan subalgebra of A.

Definition 2.19 (Weyl Groupoid, Renault [9], 4.10)**.** Let B be a sub-C^*-algebra of a C^*-algebra A. Define $dom(b) = \{x \in X : b^*b(x) > 0\}$ and $ran(b) = \{x \in X : bb^* > 0\}$. They are open subsets of X. Then $\alpha_b : dom(b) \to ran(b)$ is a unique homomorphism given $b \in$ normalizer $N(B)$. Suppose that B is abelian and B contains an approximate unit of A. Then

1. If $b \in B$, then $\alpha_b = \mathrm{id}_{dom(b)}$.

2. If $m, n \in N(B)$, then $\alpha_{mn} = \alpha_m \circ \alpha_n$ and $\alpha_{n^*} = \alpha_n^{-1}$.

Then $\mathcal{G}(B) = \{\alpha_a, a \in N(B)\}$ is a *Weyl pseudogroup* of (A, B). Define *Weyl groupoid* of (A, B) as the groupoid of germs of $\mathcal{G}(B)$. Let B be a sub C^*-algebra of a C^*-algebra A. Suppose B is abelian and B contains an approximate unit of A. The kernel of the canonical action $\underline{\alpha} : N(B) \to \mathcal{G}(B)$ is the commutant $N(B) \cap B'$ of B in $N(B)$. Additionally, if B is maximal abelian, then Ker $\underline{\alpha} = B$.

Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C_r^*(G, \Sigma)$ and $B = C_0(G^{(0)})$. Then the *Weyl groupoid* $G(B)$ of (A, B) is canonically isomorphic to G if G is topologically principal. The Weyl pseudogroup $\mathcal{G}(B)$ of (A, B) contains partial homeomorphism α_s where S is an open bisection of G such that the associated line bundle L restricted to S is trivial.

Let $D = \{(x, n, y) \in X \times N(B) \times X : n^*n(y) > 0 \text{ and } x = \alpha_n(y)\}$. Then $\Sigma(B) = D/\sim$ by the equivalent relation: $(x, n, y) \sim (x', n', y')$ iff $y = y'$ and there exist b and $b' \in B$ with $b(y), b'(y) > 0$ such that $nb = n'b'$. Then $\Sigma(B)$ has a groupoid structure over X. Given that B is MASA in A which contains an approximate unit of A, then $\mathcal{B} \to \Sigma(B) \to G(B)$ is algebraically an extension where $\mathcal{B} = \{[x, b, x] : b \in B, b(x) \neq 0\} \subset \Sigma(B)$ which is the *Weyl twist* of Cartan pair (A, B).

Remark 2.20. Let B be a Cartan subalgebra of a separable C^*-algebra A. Then there exists a twist (G, Σ) where G is a second countable locally compact Hausdorff, topologically principal étale groupoid and an isomorphism of $C_r^*(G, \Sigma)$ onto A which carries $C_0(G^{(0)})$ onto B. The twist is unique and it is isomorphic to the Weyl twist $(G(B), \Sigma(B))$.

Let (G, Σ) be a twisted étale Hausdorff locally compact second countable topologically principal groupoid. Let $A = C_r^*(G, \Sigma)$ and $B = C_0(G^{(0)})$. Then Renault [9] obtains a canonical isomorphism of extensions displayed in figure 2.2.

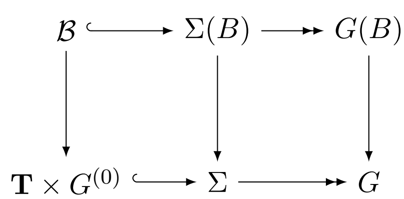

Figure 2.2: Canonical Isomorphism of Extensions.

Chapter 3

Equivalent Cartan Pairs of Renault and Feldman-Moore

In this chapter, we aim to compare the constructions of Renault and Feldman-Moore Cartan pairs. For one direction, we base our proof on the work of Li and Renault. We follow the same necessary background and assumptions as in Li and Renault [6]'s. Let G be an étale second countable locally compact Hausdorff groupoid. Let $\mathbf{T} \times G^{(0)} \overset{i}{\rightarrowtail} \Sigma \overset{j}{\twoheadrightarrow} G$ be a twist. Recall from Renault [9], every Cartan pair is of the form $(C^*_r(G, \Sigma),\, C_0(G^{(0)}))$, where G is assumed to be principal.

Let $c : G \to \Sigma$ be a Borel map such that c is a section for j so $j \circ c = id_G$. Let $X = G^{(0)}$. Let $S \subseteq G$ be an open bisection so the source and range maps restrict to homeomorphisms $r\ |_S \colon S \to r(S)$ and $s\ |_S \colon S \to s(S)$. Let α_S be the homeomorphic map $s(S) \to r(S)$, $x \to r((s\ |_S)^{-1}(x))$. Note that the existence of such Borel section was previously proven by Muhly and Williams [8]. Thus $c\ |_X = id_X$ and for all $g \in G, c(g^{-1}) = c(g)^{-1}$.

Then for every $\zeta \in \Sigma : j(\zeta c(j(\zeta))^{-1} = r(j(\zeta))$ so there is a Borel map $t : \Sigma \to T$ such that $\forall \zeta \in \Sigma$, we have $\zeta = t(\zeta)c(j(\zeta))$. Set $\sigma : G^{(2)} \to T, (g, h) \to t(c(gh)^{-1}c(g)c(h))$ where σ is a normalized cocycle. Now let μ be a $S(G)$-invariant Borel probability measure on X. Set $A = C^*_r(G, \Sigma)$ and $B = C(X)$. Then let $E : A \twoheadrightarrow B$ be the unique faithful conditional expectation. Set $\tau = \mu \circ E$. Then τ is a trace on A. Let π_τ be the GNS representation of A attached to τ and denote $\mathcal{H}_\pi$ as the underlying Hilbert space. Let R be the equivalence relation on X corresponding to G so that $R = \{(r(g), s(g)) \in X \times X : g \in G\}$.

Recall that Renault [9] proves two theorems below:

Theorem 3.1. Let (G, Σ) be a twisted étale Hausdorff locally compact second countable topologically principal groupoid. Then $C_0(G^{(0)})$ is a Cartan subalgebra of $C^*_r(G, \Sigma)$.

Theorem 3.2. Let B be a Cartan subalgebra of a separable C^*-algebra A. Then

1. there exists a twist (G, Σ) where G is a second countable locally compact Hausdorff, topologically principal étale groupoid and an isomorphism of $C^*_r(G, \Sigma)$ onto A carrying $C_0(G^{(0)})$ onto B;

2. the above twist is unique up to isomorphism; it is isomorphic to the Weyl twist $(G(B), \Sigma(B))$.

Theorem 3.3. $\pi(A)'' \cong M(R, \sigma)$ and $\pi(B)'' \cong L^\infty(X, \mu)$ where the cocycle σ is a normalized cocycle.

(Proof) To begin our proof, first notice that $M(R, \sigma)$ is the von Neumann algebra constructed in Feldman-Moore II [4] and $\pi(A)''$ is the von Neumann algebra generated by $\pi(A)$ in $L(\mathcal{H})$.

Additionally there is an isomorphism of Borel groupoids $\mathbf{T} \times_\sigma G \to \Sigma$, $(z, g) \to zc(g)$ with an inverse defined by $\zeta \to (t(\zeta), j(\zeta))$. Then we can define the multiplication of the groupoid $\mathbf{T} \times_\sigma G$ as $(z_1, \gamma_1)(z_2, \gamma_2) = (z_1 z_2 \sigma(\gamma_1, \gamma_2), \gamma_1\gamma_2)$.

Let ν be the right counting measure of μ as in Fedman-Moore I [3]. Since we have a Borel isomorphism $G \cong R, g \to (r(g), s(g))$, then $L^2(R, \nu) \cong L^2(G, \nu)$.

Define $X_i \times X_j = \{x_i \times x_j \in X_i \times X_j s.t. \mid R^{x_i \times x_j} \mid = i \times j\}$. Since all R_i's are Borel subsets of R, then $X_i \times X_j$ are Borel subsets of R. Choose $f \in C^*_r(G, \Sigma)$ and consider ν as the measure on R given by

$\int_R f d\nu = \int_{X_i \times X_j} \sum_{\gamma \in R^{x_i \times x_j}} f(\gamma) d\mu(x_i \times x_j)$. The map $C_c(G, \Sigma) \to L^2(R, \nu)$ with $f \to f \circ c$ extends to a unitary $U : \mathcal{H}_\pi \cong L^2(R, \nu)$. Since $\langle \pi(f)\xi \rangle(\gamma) = f(\gamma, \gamma)\xi(\gamma)$, then $(U\pi(f)U^*\xi)(\gamma) = f(r(\gamma), s(\gamma))\xi(\gamma)$.

Let $R = \bigcup_{i=1}^\infty R_i$ which are countably many open bisections. Observe that $\nu \mid_{R_i}$ is the pushforward of $\mu \mid_{(r(R_i), s(R_i))}$ with an inverse given by $(r(R_i), s(R_i)) \leftarrow R_i$. Let $I_{i \times j} = \{1, \cdots, i \times j\}$ for $i, j \in \mathbf{N}$. When $i = j = \infty$, $I_{i \times j} = \mathbf{N}$. We obtain a canonical unitary $V : L^2(R, \nu) \to \bigoplus_i L^2(R_i, \nu)$. Thus

$$L^2(R, \nu) \cong \bigoplus_i L^2((r(R_i), s(R_i)), \mu \mid_{(r(R_i), s(R_i))}) \text{which implies}$$

$$L^2(R,\nu) \cong \bigoplus_{i,j} L^2(X_i \times X_j, \mu\mid_{X_i\times X_j}) \otimes l^2(I_{i\times j})$$

.

Let $M_{i\times j}$ be the canonical representation of $L^\infty(X_i \times X_j, \mu\mid_{X_i\times X_j})$ on $L^2(X_i \times X_j, \mu\mid_{X_i\times X_i}$). Notice that for each $f \in C_r^*(G,\Sigma)$, we have $V\pi(f)V^* = (M_{i\times j}(f) \otimes id_{l^2(I_i\times I_j)})_{i,j}$. Therefore $\pi(C_r^*(G,\Sigma))'' = \prod_{i,j} L^\infty(X_i \times X_j, \mu \mid_{X_i\times X_j})$ which implies $\pi(C_r^*(G,\Sigma))'' \cong L^\infty(X \times X, \mu)$. Thus $\pi(A)'' \cong M(R,\sigma)$.

Similar to our proof above and following Li-Renault work, we obtain $\pi(B)'' \cong L^\infty(X,\mu)$.

□

For the other direction, notice that Feldman Moore [4] show the following:

Theorem 3.4. Let A be a Cartan subalgebra of a von Neumann algebra M on a separable Hilbert space $\mathcal{H}$. Then there exist a countable standard measure equivalence relation R on (X,μ), a $\sigma \in Z^2(R,\mathbf{T})$ and an isomorphism of M onto $W^*(R,\sigma)$ carrying A onto the diagonal subalgebra $L^\infty(X,\mu)$. The twisted relation (R,σ) is unique up to isomorphism.

Since von Neumann algebras and C^*-algebras are different, perhaps it is very technical and tedious to directly show that Felman-Moore Cartan pair is equivalent to Renault Cartan pair. Renault [9] uses étale second countable locally compact Hausdorff groupoid to build his Cartan pairs from C^*-algebras and Fell bundles. Our proof above suffices to imply that Feldman-Moore Cartan pair is isomorphic to the double commutant of Renault Cartan pair's representations.

Chapter 4

Equivalent Cartan Pairs of Donsig et al. and Feldman-Moore

During this chapter, we want to discuss equivalent constructions of Feldman-Moore and Donsig et al. Cartan pairs. We first summarize Donsig et al. theorems, corollary, and proposition as follows.

Theorem 4.1 (Donsig et al. [2], 3.8)**.** For $i = 1, 2$, suppose $(\mathcal{M}_i, \mathcal{D}_i)$ are Cartan pairs with associated extensions

$$\mathcal{P}_i \hookrightarrow \mathcal{G}_i \overset{q_i}{\to} \mathcal{S}_i.$$

Then $(\mathcal{M}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{D}_2)$ are isomorphic Cartan pairs iff their associated extensions are equivalent. Furthermore, when the extensions are equivalent and $(\mathcal{M}_i, \mathcal{D}_i)$ are in standard form, the isomorphism is implemented by a unitary operator.

Corollary 4.2 (Donsig et al. [2], 4.18)**.** Let $\pi : \mathcal{D} \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of $\mathcal{D}$ on the Hilbert space $\mathcal{H}$. Then $\lambda_\pi = \pi_* \circ \lambda$ is a representation of $\mathcal{G}$ by partial isometries on $\mathcal{U} \otimes_\pi \mathcal{H}$. If π is faithful, then λ_π is one-to-one.

Let π be the representation of $D(R, c)$ on $\mathcal{H} = L^2(X, \mu)$ as multiplication operators: for $f \in D(R, c)$, $\xi \in L^2(X, \mu)$ and $x \in X$, $\langle \pi(f)\xi \rangle (x) = f(x, x)\xi(x)$. Then π is a faithful and normal representation of $D(R, c)$. The authors show that the representation λ_π of $\mathcal{G}$ on $\mathcal{B}(\mathcal{U} \otimes_\pi \mathcal{H})$ is unitarily equivalent to the identity representation of $\mathcal{G}$ on $L^2(R, \nu)$.

Proposition 4.3 (Donsig et al. [2], 5.10)**.** The subalgebra $\mathcal{D}_q$ is a MASA in $\mathcal{M}_q$.

Theorem 4.4 (Donsig et al. [2], 5.11)**.** The pair $(\mathcal{M}_q, \mathcal{D}_q)$ is a Cartan pair.

Theorem 4.5 (Donsig et al. [2], 5.12)**.** The extension associated to the Cartan pair $(\mathcal{M}_q, \mathcal{D}_q)$ is equivalent to the extension

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\to} \mathcal{S}$$

from which $(\mathcal{M}_q, \mathcal{D}_q)$ was constructed. Furthermore, the isomorphism class of $(\mathcal{M}_q, \mathcal{D}_q)$ depends only upon the equivalence class of the extension (not on the choice of representation π or section j).

To obtain a Cartan pair from an extension, Donsig et al. [2] construct a representation of the Cartan inverse monoid where $E : \mathcal{M} \to \mathcal{D}$ is the conditional expectation and π is a representation of $\mathcal{D}$ on $\mathcal{H}$. The authors use an operator-valued reproducing kernel Hilbert space because the inverse semigroup has no linear structure compared to the von Neumann algebra $\mathcal{M}$. Donsig et al. use the order structure of $\mathcal{S}$ which arises from the action of the idempotents of $\mathcal{S}$ to build the corresponding reproducing kernel. The authors characterize the $\mathcal{D}$-bimodule of von Neumann algebra $\mathcal{M}$ instead of using appropriate subsets of Feldman-Moore 2-cohomology equivalence relation R.

Let $\mathcal{D}$ be a MASA in the von Neumann algebra $\mathcal{M}$. Starting from an inverse semigroup $\mathcal{G}$, Donsig et al. [2] construct $\mathcal{G}$ as an extension of $\mathcal{S}$ by $\mathcal{P}$ by letting $\mathcal{G} = \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $\mathcal{P} = \mathcal{G} \cap \mathcal{D}$ be the set of all partial isometries in $C(\widehat{\mathcal{E}(\mathcal{S}_i)})$ where $\widehat{\mathcal{E}(\mathcal{S}_i)}$ is the character space, and $\mathcal{S}$ be a Cartan (Boolean) inverse monoid obtained from the Munn congruence. Recall that the fundamental inverse semigroup S is the quotient of $\mathcal{G}$ by R_M. Note that the extension below

$$\mathcal{P} \overset{\iota}{\hookrightarrow} \mathcal{G} \overset{q}{\to} \mathcal{S}$$

is *trivial* if there exists a semigroup homomorphism $j : \mathcal{S} \to \mathcal{G}$ such that $q \circ j = id|_{\mathcal{S}}$.

Donsig et al. [2] first construct an extension from a given Cartan pair. Then the authors construct another Cartan pair from their extension and build another extension based on their new Cartan pair. By Theorem 4.1, Donsig et al. show their two Cartan pairs are unique by proving that these two Cartan pairs are isomorphic to each other. By theorems 4.4, the authors show that their Cartan pair exists with the assumption that $\mathcal{M}_q = (\lambda_\pi(\mathcal{G}))''$ and $\mathcal{D}_q = (\lambda_\pi(\mathcal{E}(\mathcal{G})))''$ given that $\lambda_\pi : \mathcal{G} \to \mathcal{B}(\mathcal{U} \otimes_\pi \mathcal{H})$ is the representation of $\mathcal{G}$ by partial isometries. Note that Lausch [5] shows that elements of a 2-cohomology group can parametrize equivalence classes of extensions of inverse semigroups. By the method Donsig et al. [2] construct their Cartan pair, it suffices to conclude that Feldman-Moore Cartan pair is equivalent to their Cartan pair.

Chapter 5

Conclusion and Extensions

Feldman and Moore [4] derive their Cartan pair from two-cohomology class which is very technical to prove. They have to go through fifteen propositions to complete their proof from their construction of the von Neumann algebra $M(R, \sigma)$ This motivates Donsig et al. to construct their Cartan pairs which are conceptually simpler by using the extension of inverse semigroup. Similarly, Renault constructs his Cartan pairs from C^*-algebras and étale groupoids. Our proof and discussions from chapters 3 to 5 suffice to conclude that the constructions of Feldman-Moore, Renault, and Donsig et al. Cartan pairs are equivalent.

For future work, we think of applying some of Matsnev and Resende's main proofs to conclude that Cartan pairs derived by using the extension of inverse semigroups are equivalent to those derived by using étale groupoid. First we provide theorems that Matsnev and Resende [7] show below.

Theorem 5.1 (Matsnev and Resende [7], 2.9)**.** Let G be an étale groupoid and let ρ_G : $\mathcal{I}(G) \to \mathcal{I}(G_0)$ be its full representation. Then Germs $(\mathcal{I}(G), \rho_G) \cong G$.

Theorem 5.2 (Matsnev and Resende [7], 5.3)**.** Let G be an étale groupoid with unit space X. Then $(\mathcal{I}(G), \rho_G)$ is a complete inverse semigroup over X. Any complete inverse semigroup over X arises in a similar way from an étale groupoid with unit space X.

Remark 5.3. A complete inverse semigroup over a unit space X is a complete inverse semigroup $\mathcal{S}$ equipped with a full representation $\mathcal{S} \to \mathcal{I}(X)$.

By Matsnev and Resende's proofs, we want to show the diagram in figure 5.1 commutes.

After going through Thomsen's work on semi-étale groupoids, we think that it is possible to construct another equivalent construction of Cartan pairs by relaxing the assumption of

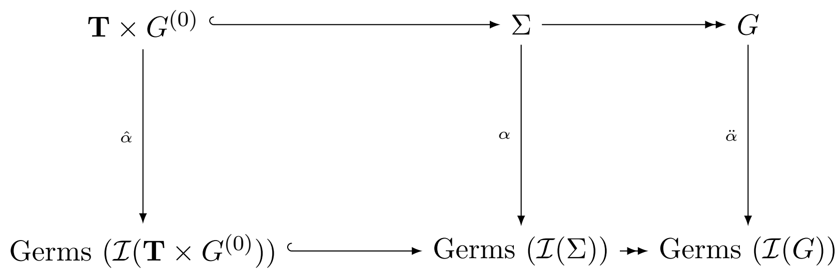

Figure 5.1: Canonical Isomorphisms of Equivalent Extensions.

étale groupoid (local homeomorphism) to semi-étale groupoid (locally injective).

Many researchers are concerned with the existence and uniqueness of Cartan pairs. Renault and Li [6] extend Renault's work to show the existence of Cartan pairs but they fail to show the uniqueness. Thus proving the uniqueness of Renault's Cartan pairs can be considered as an extension.

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