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A Survey of Line Graphs and Hamiltonian Paths

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A SURVEY OF LINE GRAPHS AND HAMILTONIAN PATHS

by

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Contents

List of Figures

Abstract

A SURVEY OF LINE GRAPHS AND HAMILTONIAN PATHS

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In this paper, we are going to explore a survey of line graphs and hamiltonian paths. Research concerning line graphs and hamiltonian paths started in the 1960's. We will investigate some recent theorems and proofs covering this topic. At the end, we will prove a main result involving line graphs and hamiltonian paths.

1 Definitions

We begin with a few definitions and examples before proceeding to the major lemmas and theorems. A graph is a representation of vertices and edges. Specifically, vertices are points, and we use $V(G)$ to represent the vertices of a graph G. Edges are ordered pairs of vertices, and we say that $E(G)$ represents the set of edges of G. Graphs can be directed or undirected, but in this paper, we will focus on undirected graphs. We can consider the graph G in Figure 1.

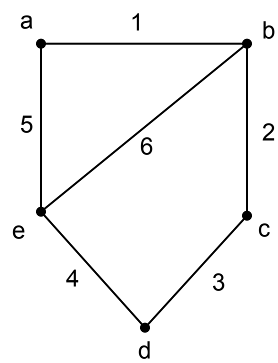

Figure 1: Graph G

Graph G has five vertices and six edges. We write $E(G) = \{1, 2, 3, 4, 5, 6\}$ and $V(G) = \{a, b, c, d, e\}$. In this paper, we may have graphs with just the vertices labeled. In this case, to name an edge, we list the two corresponding vertices together. For instance, in Figure 1, edge 1 is the same as edge ab.

The *degree* of a vertex is the number of edges incident to that vertex. In other words, we count the number of edges with that particular vertex as an end-vertex. We label the degree as $d(v)$ for a vertex v. Similarly, the *minimum degree* of a graph is defined as $\delta(G) = min\{d(v) : v \in V\}$ for a graph G. Referring back to

Figure 1, we see that $\delta(G) = 2$, since the vertex c only has two edges incident to it.

Now, two vertices are *connected* if there is a sequence of edges from one vertex to the other. The next two definitions are important in the background of hamiltonian paths and line graphs.

Definition 1.1. A graph is *k-connected* if the graph remains connected when we delete fewer than k vertices from the graph.

From Figure 1, the graph is considered 2-connected. It stays connected if we delete fewer than two vertices. We can also say that it is 1-connected.

Definition 1.2. The *connectivity* (or vertex-connectivity) of a graph is the largest k for which the graph is k-connected.

We use $\kappa(G)$ to denote the connectivity of G. From Figure 1, the connectivity is two. Removing vertices c and e will disconnect the graph, so we can write this as $\kappa(G) = 2$.

Definition 1.3. The *edge connectivity* (or *line connectivity*) of a graph is the largest k for which the graph is still connected when fewer than k edges are deleted.

We use $\lambda(G)$ to denote the edge-connectivity of G. Going back to the previous example, we have $\lambda(G) = 2$. We can remove edges 2 and 3 to disconnect the graph.

Definition 1.4. Given a graph $G = (V, E)$, the *line graph* of G, denoted $L(G)$, is a graph where $V(L(G)) = E$ and (a, b) is an edge in $L(G)$ if and only if the edges a and b in G share a common end vertex.

In order to create $L(G)$ from a given graph G, we begin by drawing a vertex in every place that G has an edge. Then, we connect the vertices in $L(G)$ based on whether the corresponding edges share a common end vertex in G. Figure 2 shows what the line graph looks like for the previous graph in Figure 1.

Notice that the edge labeled 1 in the graph G became a vertex in $L(G)$. Since 1 shared a common vertex with edges 2, 5, and 6 in G, it will be adjacent to 2, 3, and 6 as vertices in $L(G)$. Figure 2 shows how we would derive the line graph from the graph. The original graph has solid edges, while the line graph is composed of dotted edges.

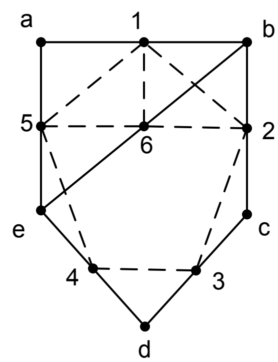

Figure 2: G and L(G)

Our next definition provides the basis for hamiltonian paths.

Definition 1.5. For a graph $G = (V, E)$, a *path* is a sequence of vertices, written $a_1, a_2, ..., a_n$ such that between any two vertices a_i and a_{i+1} there exists an edge $a_i a_{i+1}$ in the graph. A path with no repeated vertices is called a *simple path*.

Normally, a path is understood to be simple, and in this paper, we follow that convention. If we refer back to Figure 1, we can see that a, b, c, d is a path.

This brings us to one of our main definitions. A *hamiltonian path* is a path that contains every vertex in the graph exactly once. Consider Figure 1. This graph has a few hamiltonian paths; in particular, a, b, c, d, e is a hamiltonian path. Also, a, b, e, d, c is a hamiltonian path.

An Euler path is very similar to a hamiltonian path, with one key difference. An *Euler path* is a path in a graph that contains every edge exactly once. With an Euler path, we can list the vertices more than once. For instance, an Euler path in Figure 1 is b, c, d, e, a, b, e, since we've included every edge of the graph in the sequence.

For this next definition, we need to discuss what a claw graph looks like. It essentially looks like a claw, having three edges incident to one vertex, as shown in Figure 3. A *claw-free graph* is a graph that does not have a claw as an induced subgraph. In other words, for any subset of four vertices, one vertex is not adjacent to the other three vertices.

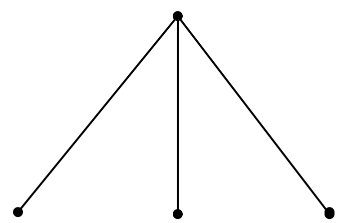

Figure 3: Claw graph

Basically, if we are able to find a part of the graph with one vertex adjacent to three vertices as in Figure 3, then the graph is not claw-free.

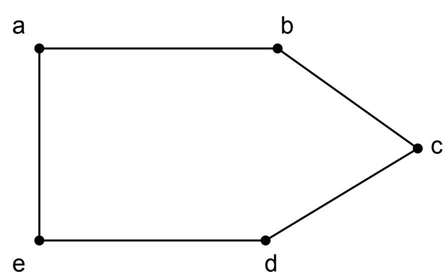

Figure 4: Claw-free graph

Figure 4 is a claw-free graph, while the graph in Figure 5 is not. The claw is in bold in the figure.

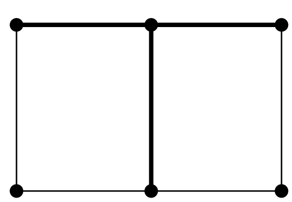

Figure 5: Graph with a claw as an induced subgraph

We now consider two similar, but subtly different definitions. A *cycle* is a closed path that starts and ends with the same vertex. A *circuit* is a cycle with no repeated vertices, except for the first and last vertex.

Essentially, a circuit is a cycle, but not vice versa. Again, consider Figure 1. A cycle in this graph is a, b, e, a, b, e, a, while a circuit would just be a, b, e, a.

Now, we can also consider a hamiltonian cycle. A *hamiltonian cycle* has the same characteristics as a hamiltonian path, but it is a cycle instead of a path. From Figure 1, a hamiltonian cycle is a, b, c, d, e, a.

Now, we can consider a trail. A *trail* is a sequence of alternating vertices and edges such that the edges are distinct. From Figure 1, $a, 1, b, 2, c, 3, d$ is a trail. We will discuss special types of trails throughout this paper. These next few definitions are taken from the paper "On Hamiltonian Line Graphs and Connectivity." [12]

Definition 1.6. A trail is a *dominating trail*, denoted xT_dy, where x is the first vertex and y is the last vertex of the trail, if each edge of a graph G is incident with at least one internal vertex of the trail.

In Figure 1, if we choose vertices a and c, we claim that $aT_dc = a, 1, b, 6, e, 4, d,$ $3, c$ is a dominating trail. Notice that the internal vertices are b, e, and d. First, notice that edge 1 is incident with b. Edge 2 is incident with b. Then, edge 3 is connected to d. Edge 4 is incident with d and e. Next, we have that edge 5 is incident with e. Finally, edge 6 is connected to b and e. So, aT_dc is a dominating trail.

Similarly, we can define spanning trail.

Definition 1.7. A *spanning trail* is a dominating trail that contains all vertices of a graph G, and it is denoted xT_sy, where x and y are the beginning and end vertices of the trail, respectively.

We can check to see if aT_sc is a spanning trail as well. It is a dominating trail, so we just need for it to contain all of the vertices of the graph: a, b, c, d, e. Notice that $aT_sc = a, 1, b, 6, e, 4, d, 3, c$ does indeed contain all of the vertices, so it is a spanning trail.

With these two definitions in mind, we can specify graphs that have these trails as follows.

Definition 1.8. A graph G is *dominating trailable* if, for each pair of vertices x and y of G, there exists a dominating trail xT_dy with end-vertices x and y.

The graph from Figure 1 is not dominating trailable. If we choose vertices c and d, then in order to include edge 3, we have to include either edge 2 or 4 twice in the trail.

Definition 1.9. A graph G is *spanning trailable* if there exists a spanning trail xT_sy with end-vertices x and y, for all x and y in $V(G)$.

We can use the same two vertices, c and d, to disprove this. Since there wasn't a dominating trail between these two, there isn't a spanning trail. So, this graph is not spanning trailable.

Definition 1.10. A graph G is *hamiltonian-connected* if for each pair, u and v, of vertices of G, there exists a hamiltonian path with end-vertices u and v.

For Figure 1, there is not a hamiltonian path between b and d. In order to include c, we would have to include b or d twice, violating the definition of hamiltonian path. So, graph G is not hamiltonian-connected.

We will be interested in graphs that are both dominating trailable and spanning trailable. Consider Graph P in Figure 6.

To show that P is dominating trailable, we need a dominating trail between every pair of edges in P. However, we won't need to check every pair. Notice that the graph is symmetric. So, if we check vertex a with every edge, then we won't

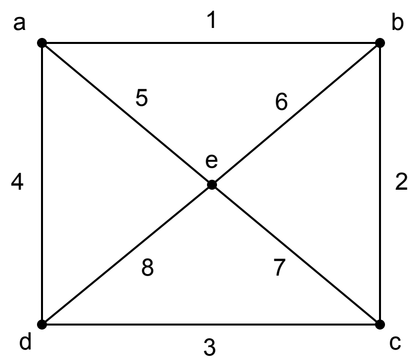

Figure 6: Graph P

need to check vertices b, c, and d (just think of rotating the graph to match these other three edges up with a). Then, we would just need to check vertex e, and we would be finished. One final thing we can do is to specifically find spanning trails. With this, we can show that the graph is both spanning trailable and dominating trailable.

We can proceed by starting with vertex a. Consider vertices a and b. Then, a spanning trail is $aT_sb = a, 1, b, 6, e, 8, d, 3, c, 2, b$. Notice that the internal vertices are b, e, d, and c. Every edge in the graph is incident with one of these vertices, so aT_sb is a spanning trail. For vertices a and c, we have $aT_sc = a, 1, b, 6, e, 8, d, 3, c$. Next, we look at a and d. A spanning trail is $aT_sd = a, 4, d, 8, e, 6, b, 2, c, 3, d$. Finally, notice that between a and e, $aT_se = a, 1, b, 2, c, 3, d, 4, a, 5, e$.

Now, we can check all of the vertices with e. Notice that we already checked a and e together. We can just reverse the path to check with e and a with $eT_sa = e, 5, a, 4, d, 3, c, 2, b, 1, a$. So, consider e and b. Then, $eT_sb = e, 6, b, 2, c, 3, d, 4, a,\ 1, b$. Then, for e and c, we have $eT_sc = e, 7, c, 3, d, 4, a, 1, b, 2, c$. Now, looking at e and d gives $eT_sd = e, 8, d, 4, a, 1, b, 2, c, 3, d$.

So, we have found a spanning trail for each set of vertices, defining trails for b, c, and d similar to what we did for a. Therefore, graph P is both dominating trailable and spanning trailable.

Also, notice that this graph is hamiltonian-connected. We can check each set of vertices and find a hamiltonian path. As with the spanning trail above, by finding hamiltonian paths with a, we know that there will be hamiltonian paths with b, c, and d as well.

First, we check each vertex with a. Between a and b, a hamiltonian path is a, d, e, c, b. For a and c, we have a, b, e, d, c. With a and d, one hamiltonian path is a, b, e, c, d. Finally, for a and e, we can see that a, b, c, d, e is a hamiltonian path.

Next, we can look at e. For e and a, we can use the path from above to get e, d, c, b, a. For e and b, a hamiltonian path is e, c, d, a, b. With e and c, we have e, d, a, b, c. Finally, given e and d, e, a, b, c, d is a hamiltonian path.

By showing that each set of vertices have a hamiltonian path between them, we have that graph P is hamiltonian-connected.

Definition 1.11. A *component* is a maximal connected subgraph that is not properly contained in any connected subset of vertices of the original graph.

We will use $\omega(G)$ to denote the number of components of a graph G. We can consider graph C in Figure 7. This graph consists of two components.

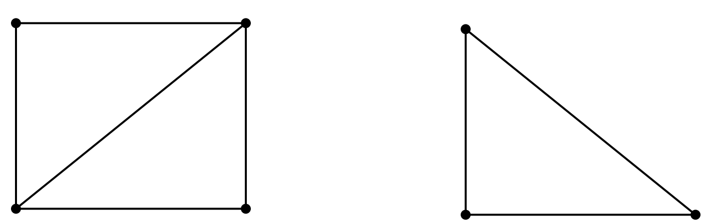

Figure 7: A graph C containing components

Now, we can define the procedure for finding the reduced graph of a given graph.

Definition 1.12. $R(G)$ is called a *reduced graph* of G, and is created by applying these two operations as needed:

- Operation $R1$: (deletion) delete a vertex, which has degree at most 3 but is adjacent to at most one vertex, and delete its incident edges.
- Operation $R2$: (contraction) delete a vertex u with degree 2 and its incident edges uv and uw while $v \neq w$ and add a new edge vw.

To illustrate this definition, we will find the reduced graph of the following graph Q.

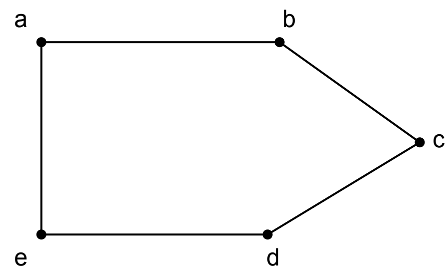

Figure 8: A simple graph, Q

First, notice that we can use operation $R2$ on any vertex in this graph, since each vertex has degree two. So, we will just start with the vertex c. If we delete that vertex, then we also delete the edges connected to it. Also, we have to draw a new edge between b and d. So, we now have the graph on the left in Figure 9.

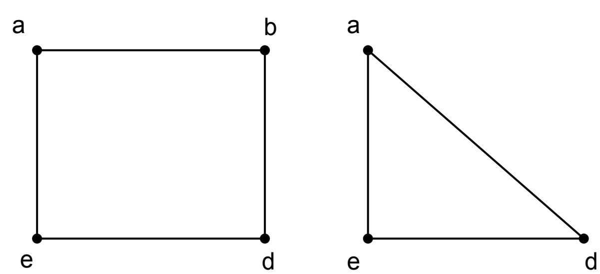

Figure 9: First and second step using $R2$

Since we still have vertices of degree two, we will continue to use operation $R2$. We remove vertex b next and create a new edge between a and d to get the next graph in Figure 9.

We still have a vertex of degree two, so $R2$ still applies. We remove vertex a and create an edge between d and e to get the first graph in Figure 10. Finally, we apply $R2$ one more time to vertex d by removing d and creating an edge between e and e. We now have the reduced graph of Q, or $R(Q)$.

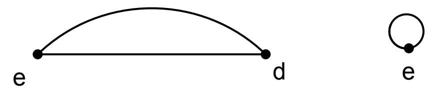

Figure 10: Third and fourth step using $R2$

We can also consider a different example. With Graph D in Figure 11, we can apply $R1$ first and delete vertex a. Also, we can apply $R2$ to delete vertex e. Then, we will have the reduced graph of D in Figure 11.

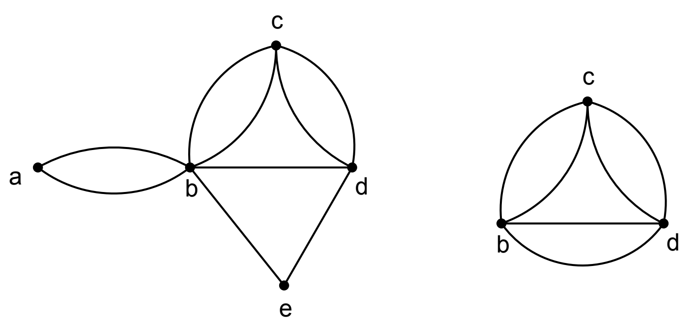

Figure 11: Graph D and its reduced graph $R(D)$

2 Background for Hamiltonian Paths and Line Graphs

We now consider various papers covering the topics of line graphs and hamiltonian paths. We also discuss two important conjectures that have yet to be proven, but would provide a very strong condition for a line graph to have a hamiltonian path. There were a few small papers that briefly mentioned line graphs and hamiltonian paths together before 1965, but the first major paper about these two topics was titled, "On Eulerian and Hamiltonian Graphs and Line Graphs," written by Harary and Nash-Williams [3] in 1965. Most papers published about hamiltonian paths and line graphs since the sixties reference to this paper and mention the basics. In particular, we list four main propositions from the paper below.

Proposition 2.1. If G is hamiltonian, then $L(G)$ is hamiltonian.

This is a nice, basic result to see if a line graph is hamiltonian. A graph is hamiltonian if there exists a hamiltonian cycle in the graph. It may be easier to find a hamiltonian cycle in G than $L(G)$, but from this proposition, we would get that $L(G)$ is hamiltonian.

Proposition 2.2. $L(G)$ is eulerian if and only if the degrees of the vertices of G are all of the same parity.

A graph is eulerian if there exists an Euler path in the graph. This proposition requires that the degrees of all of the vertices in G be of the same parity for $L(G)$ to be eulerian.

Proposition 2.3. $L(G)$ is hamiltonian if and only if there is a cycle in G which includes at least one end-point of each line of G.

In this case, the line graph of G has a hamiltonian cycle, which consists of a cycle containing every vertex exactly once in this graph. If we consider graph G, we can say that this cycle contains every edge of G exactly once.

Proposition 2.4. $L2(G)$ is hamiltonian if and only if there is a spanning cycle in G.

In this proposition, $L2(G)$ refers to the line graph of the line graph of G, or $L(L(G))$. A spanning cycle is a cycle containing all of the vertices of the graph. So, if we can find a cycle with all of the vertices, then we should be able to find a hamiltonian path.

The next paper we consider has an important theorem used for the connectivity of line graphs and was written by Chartrand and Steward [1] in 1969. They proved the following theorem.

Theorem 2.1. *If a graph G is m-line connected, $m \geq 2$, then its line graph $L(G)$ is m-connected.*

We mentioned in Definition 1.3 that line connectivity is another term for edge connectivity. Theorem 2.1 is considering a graph that has edge connectivity of m. We know that the edges of G correspond to the vertices of $L(G)$, so Theorem 2.1 relates the edge connectivity of G to the vertex connectivity of $L(G)$. This is helpful for getting an idea of the connectivity of a line graph without actually having to draw it out when dealing with line graphs and hamiltonian paths.

Significant results concerning line graphs and hamiltonian paths did not appear in any papers for about fifteen years. The next one was written in 1984 by Matthews and Sumner [6]. They conjectured the following.

Conjecture 2.1. Every 4-connected claw-free graph is hamiltonian.

This statement focused on a claw-free graph instead of a line graph, but we did get that the graph is hamiltonian. Consider the 4-connected claw-free graph of K_5 in Figure 12. The conjecture claims that this graph is hamiltonian. Indeed, a hamiltonian cycle is $1, 2, 3, 4, 5, 1$, so the graph is hamiltonian.

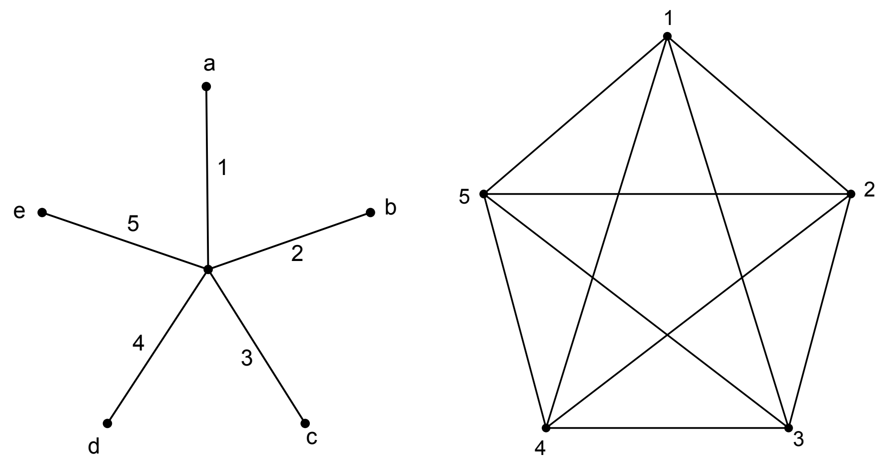

Figure 12: S_5 and its line graph, K_5

Shortly afterward, in 1986, Thomassen [9] stated an important conjecture concerning hamiltonian graphs.

Conjecture 2.2. If G is a graph, such that its line graph $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.

Conjecture 2.2 claims that if the line graph of a graph is 4-connected, then the line graph is hamiltonian. By proving this conjecture, we would get a very strong result for a line graph being hamiltonian. Many people tried to prove Conjecture 2.1 and Conjecture 2.2, but the condition that the graph has to be claw-free or 4-connected was too strong. Zhan [12] weakened the hypothesis and in 1991 proved the following theorem.

Theorem 2.2. *Every 7-connected line graph is hamiltonian-connected.*

If we can show that a line graph of some graph is 7-connected, then by Theorem 2.2, we get that the line graph is hamiltonian-connected. We will focus on the process of proving Theorem 2.2 in the next section. Consider the graph in Figure 13, S_8, and its line graph, the graph of K_8.

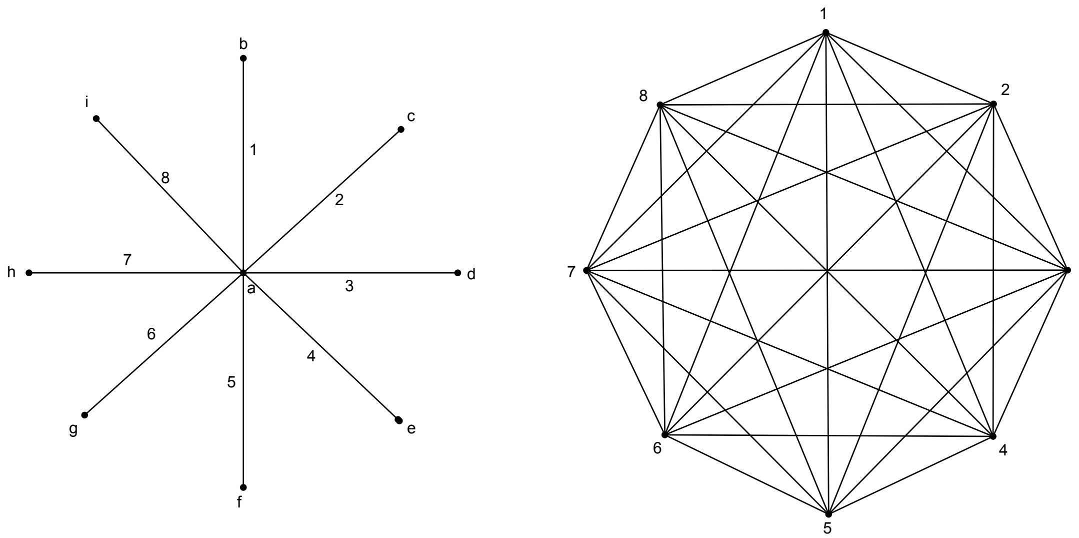

Figure 13: S_8 and its line graph, K_8

From Theorem 2.2, the graph of K_8 is hamiltonian-connected. We can see that for any two vertices, there does exist a hamiltonian path with these two endpoints. For instance, if we choose 1 and 5, then we can start with 1, go to 2, 3, 4, and instead of going to 5, go across to 8. Then, work back to 5 by going to 7, 6, and then to 5. So, the hamiltonian path is $1, 2, 3, 4, 8, 7, 6, 5$. Since we can find a hamiltonian path between any two vertices, this graph is hamiltonian-connected.

In 1997, Ryjacek [8] proved two different theorems. The first theorem looks at a claw-free graph instead of a line graph.

Theorem 2.3. *Every 7-connected claw-free graph is hamiltonian.*

Recall the graph of K_8 in Figure 13. This is the line graph of S_8, but it is also a claw-free graph. We will not get a claw with any induced subgraph, since every set of four vertices is not isomorphic to the claw graph. We already know that K_8 is hamiltonian, but this theorem provides a different, sufficient condition for a graph to be hamiltonian.

Recall Conjectures 2.1 and 2.2. Ryjacek proved in this next theorem that if one conjecture is true, the other one is also true.

Theorem 2.4. *The following two conjectures are equivalent. Every 4-connected claw-free graph is hamiltonian. Every 4-connected line graph is hamiltonian.*

This theorem simplifies the problem of having to prove Conjecture 2.1 and Conjecture 2.2. Now, only one conjecture needs to be proven, and we will get the other conjecture as a result of this theorem.

Also, we can consider the relationship between a claw-free graph and a line graph. One fact about line graphs is that line graphs are claw-free. For instance, if we take an edge in a graph and try to form a claw in the line graph, we would have to have three edges adjacent to the first edge that are not pairwise adjacent to each other. But, an edge only has two endpoints, so two of the three edges would have to be adjacent. Then, we would not get a claw in the line graph. With this fact, there is a relationship between the conditions of the two parts to this theorem.

We did not get any more results about hamiltonian paths and line graphs until 2006. A paper written by Lai, Shao, Wu, and Zhou [5] had the following theorem.

Theorem 2.5. *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

In the paper, they also listed the following corollary to this theorem.

Corollary 2.1. Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.

Line graphs fall under the category of claw-free graphs, so this corollary provides us with more graphs that are hamiltonian. Next, in 2010, we get two theorems published by M. Zhan [11] that give different conditions for a line graph to be hamiltonian.

Theorem 2.6. *Let H be a 6-connected line graph. If $d_6(H) \leq 74$, then H is hamiltonian.*

Theorem 2.7. *Let H be a 6-connected line graph. If $d_6(H) \leq 54$ or $G[D_6(g)]$ contains at most 5 vertex disjoint K_4's, then H is hamiltonian-connected.*

In these two theorems, $d_6(H)$ refers to the number of vertices of degree 6 in H. The first theorem requires that this number is less than or equal to 74 in order for H to be hamiltonian.

The most recent result concerning hamiltonian line graphs was published by Kaiser and Vrana [4] in 2011 about the following theorem.

Theorem 2.8. *Every 5-connected line graph with minimum degree at least 6 is hamiltonian.*

This theorem gives us a smaller connectivity for a line graph to be hamiltonian, but it also contains the additional restriction that the graph has a minimum degree of at least six.

Thus, although two important conjectures involving a 4-connected graphs and line graphs have not been proven, significant progress has been made in the study of line graphs and hamiltonian paths. The most recent result, Theorem 2.8, tells us that a 5-connected line graph is hamiltonian, but we still don't quite have 4-connected.

3 Proof of Theorem 2.2

We will focus on the requirements for proving that every 7-connected line graph is hamiltonian-connected. The results and proofs from this section come from the paper, "On Hamiltonian Line Graphs and Connectivity" [12]. We will first consider a few lemmas.

Lemma 3.1. Let G be a graph with at least four vertices. Then, the line graph $L(G)$ is hamiltonian-connected if and only if G is dominating trailable.

Proof. We begin by assuming $L(G)$ is hamiltonian-connected. So, between any two vertices, x and y, in $L(G)$, we have a hamiltonian path written as $x = x_0, x_1, x_2, ..., x_n = y$, where $n + 1$ is the number of vertices in $L(G)$. Since the vertices of $L(G)$ correspond to the edges of G, then $x_0, x_1, x_2, ..., x_n$ is a sequence of edges in G, where $x_i \in E(G)$ for $i = 0, 1, ..., n$. Let v_i be the common vertex between x_i and x_{i+1} in G and create a list of vertices as $v_0, v_1, ..., v_n$. Now, in this list of vertices $v_0, v_1, ..., v_n$, some vertices may appear more than once. So, create a subset $w_1, w_2, ..., w_k$, where each vertex appears only once and $k \leq n$. In creating this, once a vertex is listed, we won't list it again. Now, for two vertices w_i and w_{i+1}, where $i = 1, 2, ..., k$, list the corresponding edge between these two as e_i. Then, $x_1, w_1, e_1, w_2, e_2, ..., w_k, e_k, y$ is a dominating trail in G between edges x and y, since every edge in G is incident with one of $w_1, w_2, ..., w_k$. Since this trail works for all edges in G, we can say that G is dominating trailable.

Conversely, we can assume G is dominating trailable, and let x and y be edges of G. Then, there exists a dominating trail between x and y written as $x, v_1, x_1, v_2, ..., v_n, x_n = y$, where $x_i \in E(G)$ for all $i = 1, 2, ..., n$. So, n is the number of internal vertices of the trail. For the remaining edges not listed in the dominating trail, we will partition in the following way. Create n sets, labeled $S_1, S_2, ..., S_n$. Next, for an edge incident with v_i, place that edge in the corresponding set, S_i. Then, start this process with v_1, and once an edge is placed in a set, do not consider that edge again. Notice that some sets may be empty, and some sets may have more than one element.

Define the elements of S_i as $s_{i,1}, 1, s_{i,2}, 2, ..., s_{i,r}, r$ where r is the length of S_i. Then, consider the list $x, S_1, x_1, S_2, ..., S_n, y$ written as $x, s_{1,1}, 1, s_{1,2}, 2, ..., s_{1,r}, r, x_1, s_{2,1}, 1, ..., s_{n,r}, r, y$. Since the edges of G correspond to the vertices of $L(G)$, we now classify this sequence as a list of vertices in $L(G)$. This sequence is a path, since it consists of distinct vertices of $L(G)$ with each vertex in the list adjacent to the one before and after it. By construction, we have accounted for every edge in G, and thus every vertex in $L(G)$. This makes the path $x, s_{1,1}, 1, s_{1,2}, 2, ..., s_{1,r}, r, x_1, s_{2,1}, 1, ..., s_{n,r}, r, y$ a hamiltonian path in $L(G)$. Since this is true for any x and $y \in E(G)$, $L(G)$ is hamiltonian connected. $\square$

Recall Figure 6 with graph P. Using this graph, we will show that if the line graph of P is hamiltonian-connected, then, using Lemma 3.1, we can say that P is dominating trailable. Figure 14 shows the line graph of P.

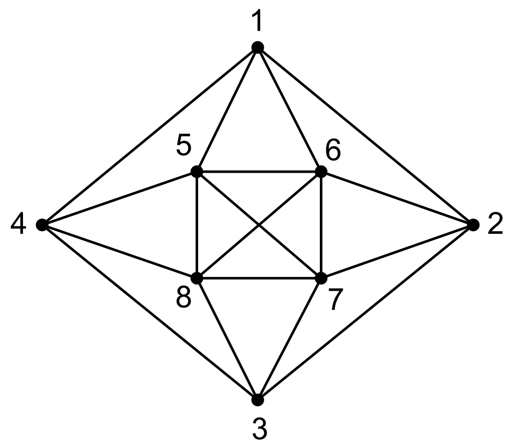

Figure 14: Line graph of graph P

To show that $L(P)$ is hamiltonian-connected, we need to find a hamiltonian path between every pair of vertices in the graph. Since the graph is symmetric, if we find a hamiltonian path between vertex 1 and all other vertices, then we will not need to find paths starting at 2, 3, and 4. Likewise, by finding hamiltonian paths starting with vertex 5, we will not have to find paths with vertices 6, 7, and 8.

For vertices 1 and 2, a hamiltonian path is $1, 4, 3, 8, 5, 6, 7, 2$. Between 1 and 3, we have $1, 2, 6, 7, 8, 5, 4, 3$. With 1 and 4, a hamiltonian path is $1, 2, 3, 7, 6, 5, 8, 4$. Next, vertices 1 and 5 have the path $1, 2, 3, 4, 8, 7, 6, 5$. For 1 and 6, we have $1, 2, 3, 4, 5, 8, 7, 6$. With 1 and 7, a hamiltonian path is $1, 2, 3, 4, 5, 6, 8, 7$. Finally, for 1 and 8, we have $1, 2, 3, 4, 5, 6, 7, 8$ as a hamiltonian path.

Now, when considering vertices 5 and 1, we can use $5, 6, 7, 8, 4, 3, 2, 1$ as the hamiltonian path. For 5 and 2, we have $5, 1, 4, 3, 8, 7, 6, 2$. Next, 5 and 3 gives us $5, 4, 1, 2, 6, 7, 8, 3$. Then, with 5 and 4, we get $5, 1, 2, 3, 7, 6, 8, 4$. For 5 and 6, a hamiltonian path is $5, 1, 2, 3, 4, 8, 7, 6$. So, 5 and 7 gives us $5, 1, 2, 3, 4, 8, 6, 7$. Finally, with 5 and 8, we have $5, 4, 3, 2, 1, 6, 7, 8$.

We have shown that the graph of $L(P)$ is hamiltonian-connected. Now, by Lemma 3.1, we can say that the graph of P is dominating trailable. We can also consider the contrapositive of Lemma 3.1 with Figures 1 and 2. In the graph of $L(G)$, there does not exist a hamiltonian path between vertices 2 and 4. Vertex 3 is never included in any hamiltonian path, so the line graph of G is not hamiltonian-connected. By the contrapositive of Lemma 3.1, we can say that G is not dominating trailable.

We now consider the next lemma and prove it.

Lemma 3.2. If G is a graph, which is not multi-star with edge-multiplicity at most 3, and if its line graph $L(G)$ has connectivity at least 4, then there is a unique graph (up to isomorphism) $R(G)$, so called reduced graph of G, obtained by applying a sequence of operations $R1$ and $R2$ from G such that:

1. $\delta(R(G)) \geq 3$;

2. $\kappa(L(R(G))) \geq \kappa(L(G))$;

3. $V(R(G))$ is a dominating set of G.

Proof. Let G be a graph with $\kappa(L(G)) \geq 4$. Define $R1$ as a deletion and $R2$ as a contraction.

Towards a contradiction, assume $\delta(R(G)) < 3$. Then, we consider the following three different cases.

For case one, we have $\delta(R(G)) = 0$. A graph with a minimum degree of zero implies that there is a vertex without any edges connected to it. Since we don't have any edges in $R(G)$ to create the line graph with, $L(G)$ does not have connectivity of four.

For case two, we have $\delta(R(G)) = 1$. Now, we have a vertex that is adjacent to one other vertex, which contradicts how $R(G)$ was created using $R1$.

For case three, we have $\delta(R(G)) = 2$. Then, a vertex is either connected to one vertex with two edges or connected to two vertices. In the first situation, this will violate how $R1$ was used to create $R(G)$. In the second situation, this will violate how $R2$ was used to create $R(G)$. Therefore, $\delta(R(G)) \geq 3$.

Now, let n represent the number of operations (number of times $R1$ and/or $R2$ is used) applied to G to create $R(G)$. We will use induction to show that the connectivity of $L(R(G))$ is greater than or equal to the connectivity of $L(G)$. For the base case, if $n = 0$, then $G = R(G)$. So, $L(G) = L(R(G))$. Thus, we have $\kappa(L(R(G)) = \kappa(L(G))$, and $\kappa(L(R(G)) \geq \kappa(L(G))$. Now, let $\kappa(L(R_{n-1}(G)) \geq \kappa(L(G))$ for $n-1$ steps, and we can show that this works for the nth step. Now, consider $R_{n-1}(G)$ and $R_n(G)$. These two graphs differ by one operation, either $R1$ or $R2$. So, we will look at each of these separately.

For the first case, $R1$ is applied to $R_{n-1}(G)$ to get $R_n(G)$. So, a vertex adjacent to only one vertex in $R_{n-1}(G)$ was deleted. So, if the vertex deleted had the minimum degree in $L(R_{n-1}(G))$, then the minimum degree of $L(R_n(G))$ will be greater than or equal to that of $L(R_{n-1}(G))$. If this vertex did not have the minimum degree, then the minimum degree will stay the same in both graphs. Either way, $\kappa(L(R_n(G)) \geq \kappa(L(R_{n-1}(G))$. From above, $\kappa(L(R_{n-1}(G)) \geq \kappa(L(G))$, so combining these gives us $\kappa(L(R_n(G)) \geq \kappa(L(G))$.

For the second case, $R2$ is applied to $R_{n-1}(G)$ to get $R_n(G)$. So, the vertex deleted had a degree of two. Again, if one of the edges deleted had the minimum degree in $L(R_{n-1}(G))$, then the minimum degree of $L(R_n(G))$ will be greater than that of $L(R_{n-1}(G))$. Otherwise, it will stay the same. So, $\kappa(L(R_n(G)) \geq \kappa(L(R_{n-1}(G))) \geq \kappa(L(G))$.

Therefore, by induction, $(L(R_n(G)) \geq (L(G))$, or for in general, $\kappa(L(R(G)) \geq \kappa(L(G))$.

Define a set D as the set of vertices in G such that the degree of each vertex is less than or equal to three and each vertex is adjacent to at least two vertices in G. First, we need to show that $D = V(R(G))$, and we show this by containment. Let

$v \in V(R(G))$. Since v is in the reduced graph, the minimum degree of v is three, using Part 1 of Lemma 3.2. Also, v is adjacent to two or more vertices, since if it weren't, we could have reduced the graph more. From how D is defined, $v \in D$.

Let $v \in D$, and towards a contradiction, assume $v \in V(R(G))$. Then, v is a vertex that was deleted from G using $R1$ or $R2$. Since $v \in D$, v has degree of at least three. So, when v was deleted, the edges incident with v at this step in the graph corresponding to the edges incident with v in G, which disconnect $L(G)$ with a cardinality of at most three. However, this contradicts $L(G)$ having connectivity of at least four. So, $v \in V(R(G))$.

Next, we need to show that $V(R(G))$ is a dominating set. Towards a contradiction, assume that it is not a dominating set. By definition of a dominating set, there exists an edge $u'v'$ that is not incident with any vertex of $V(R(G))$. Since $L(G)$ is 4-connected, then there are at least four edges connected to $u'v'$. Choose v adjacent to v' in G, where $v \in V(R(G))$. Notice that since v' and $u' \in V(R(G))$, either the degree of each is less than three or each is adjacent to only one vertex. This contradicts $L(G)$ being 4-connected. Therefore, $V(R(G))$ is a dominating set of G. $\square$

We will use Figures 6 and 14 to demonstrate Lemma 3.2. Since graph P is not a multi-star and the line graph of P has connectivity four, we meet the criteria for the lemma. Notice that graph P cannot be reduced, so $R(P) = P$ in this case. So, from Part 1, we can say that $\delta(R(P)) \geq 3$, and in particular, we see that $\delta(R(P)) = 3$. From Part 2, $\kappa(L(R(P)) \geq \kappa(L(P))$, and since $L(R(P))$ and $(L(P)$ are the same graphs, $\kappa(L(R(P)) = 4$ and $\kappa(L(P)) = 4$. Finally, with Part 3, $V(R(P)) = V(P)$ is a dominating set, and the original vertices in a graph always compose a dominating set. Next, we can consider the following lemma.

Lemma 3.3. If G is a graph, which is not a multi-star with the edge multiplicity at most 3, and, if its line graph, $L(G)$, has connectivity of at least 4, then G is dominating trailable if its reduced graph $R(G)$ is spanning trailable.

Proof. Assume that $L(G)$ has connectivity of at least four and that $R(G)$ is spanning trailable. Choose two edges of G called $x = uv$ and $y = st$. We need to choose two other edges, x' and y' in $R(G)$ defined as follows. If $x \in R(G)$, then we will say $x' = x$. Next, if vertex v has degree two and uv and vw are two edges of G with $u \neq w$, then $x' = uw \in R(G)$. Otherwise, if vertex v has degree one, then x' is incident with u in $R(G)$.

For $y' \neq x'$, we will choose y' similarly. If $y \in R(G)$, then $y' = y$. If vertex t has degree two and st and tr are two edges of G with $s \neq r$, then $y' = sr \in R(G)$. Otherwise, if vertex t has degree one, then y' is incident with s in $R(G)$

Since $R(G)$ is spanning trailable, there exists a spanning trail between x' and y', and we call it $x'T_{s*}y'$. Extend $x'T_{s*}y'$ to be just a trail in G. Now, let T be the trail in G corresponding to the trail $x'T_{s*}y'$. So, consider xTy. From Part 3 of Lemma 3.2, $V(R(G))$ is a dominating set in G. So, every edge has one vertex in $V(R(G))$. By definition of dominating trail, we can see that xTy is a dominating trail, which we denote xT_dy. Therefore, G is dominating trailable. $\square$

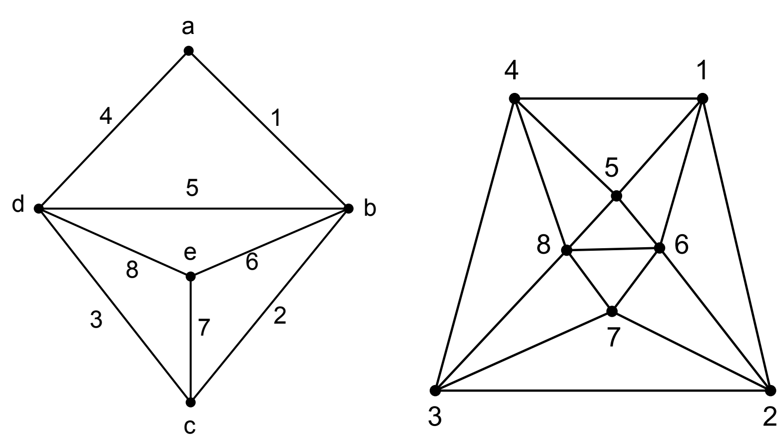

Figure 15: Graphs S and $L(S)$

We will consider Figure 15 to show an example of how Lemma 3.3 can be used. The graph from Figure 15 shows a graph S together with its line graph, $L(S)$. We can see that S is not a multi-star, and the connectivity of the line graph of S is four, which fits the criteria for the first part of Lemma 3.3.

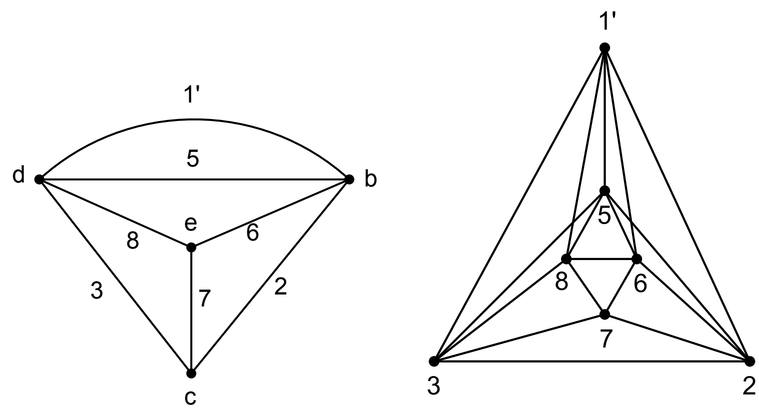

Figure 16: Graphs $R(S)$ and $L(R(S))$

So, we just need to see if $R(S)$ from Figure 16 is spanning trailable. To do that, we need to find a spanning trail between each pair of vertices. Notice that $V(R(S)) = \{b, c, d, e\}$, so we can start checking with vertex b. Between b and c, a spanning trail is $bT_sc = b, 1', d, 8, e, 6, b, 2, c$. Next, for b and d, we have $bT_sd = b, 2, c, 7, e, 8, d, 5, b, 1', d$. For b and e, a spanning trail is $bT_se = b, 2, c, 3, d, 1', b, 6, e$. Now, we can check c with d and e. With c and d, we have $cT_sd = c, 7, e, 8, d, 5, b, 1', d$. For vertices c and e, the spanning trail is $cT_se = c, 7, e, 6, b, 1', d, 8, e$. Finally, checking d and e, we have $dT_se = d, 3, c, 2, b, 5, d, 8, e$. For each pair of vertices, we have a spanning trail for the graph of $R(S)$. By Lemma 3.3, S is dominating trailable.

For the next lemma that will be used in the proof of the main result, we need to define a few graphs. Assume that there is a graph G with $\delta(G) = 3$ and $\kappa(L(G)) \geq 7$ and a graph S that is a subset of $E(G)$. We let $G_1, ..., G_r, G_{r+1}, ..., G_t, G_{t+1}, ..., G_\omega$ be the components of G/S, for $1 \leq r \leq s \leq \omega = \omega(G - S)$. In other words, $G_1, ..., G_r$ are the components consisting of a single vertex of degree three. This gives $G_{r+1}, ..., G_t$ as the components with at least one vertex adjacent to some vertices of $\bigcup_1^r V(G_i)$. $G_{t+1}, ..., G_\omega$ are the remaining components.

Note that each set of components may be empty. For a subgraph H of G, we let

$M(H)$ be a subset of $E(G)$ with exactly one end-vertex in $V(H)$. Also, let $m(H)$ be the cardinality of $M(H)$.

Lemma 3.4. If $\omega(G/S) \geq 3$ for a subset S of the edge set $E(G)$, then

1. $m(G_i) = 3$, for $1 \leq i \leq r$;
2. $m(G_i) \geq 6$, for $r+1 \leq i \leq t$;
3. $m(G_i) \geq 4$, for $t+1 \leq i \leq \omega$;
4. $\sum_{r+1}^{t} m(G_i) \geq \sum_{1}^{r} m(G_i)$;
5. $\bigcup_{1}^{\omega} M(G_i) \subseteq S$.

Proof. Assume that $\delta(G) \geq 3$, $\kappa(L(G)) \geq 7$, and $\omega(G-S) \geq 3$. So, we can focus on one part at a time.

Notice that each $G_1, ..., G_r$ consists of a single vertex of degree three. So, there are three edges of G with exactly one end-vertex incident with this single vertex. Thus, $m(G_i) = 3$.

We know that the minimum degree of G is three, so $m(G_i) \geq 3$. We already defined G_i for $1 \leq i \leq r$ to contain all of the vertices of degree three, so for $t+1 \leq i \leq \omega$, $m(G_i) \geq 4$.

For $1 \leq i \leq r$, each G_i is composed of a single vertex of degree three. Since, $\kappa(L(G)) \geq 7$, each edge of this component has to be adjacent to at least five other edges. For $r+1 \leq i \leq t$, G_i is a component with at least one vertex adjacent to one of the vertices of $\bigcup_{1}^{r} V(G_i)$. So, for each edge adjacent to at least five other edges, there will be one vertex of $\bigcup_{1}^{r} V(G_i)$ adjacent to at least five in $\bigcup_{r+1}^{t} V(G_i)$. Therefore, $\sum_{r+1}^{t} m(G_i) \geq \sum_{1}^{r} m(G_i)$.

By definition, we stated that $\sum_{1}^{\omega} M(G_i)$ is the sum of the set of all edges of G that have exactly one end-vertex in $V(G_i)$. Each G_i was formed by removing a part of S from G, where the edge of S had one end-vertex in G_i. So, the set of all of these edges form a subset of S. Thus, $\bigcup_1^{\omega} M(G_i) \subseteq S$.

$\square$

Because the conditions that $\delta(g) \geq 3$ and $\kappa(L(G)) \geq 7$ are fairly strong, we show that there are graphs having these properties by considering Figure 17.

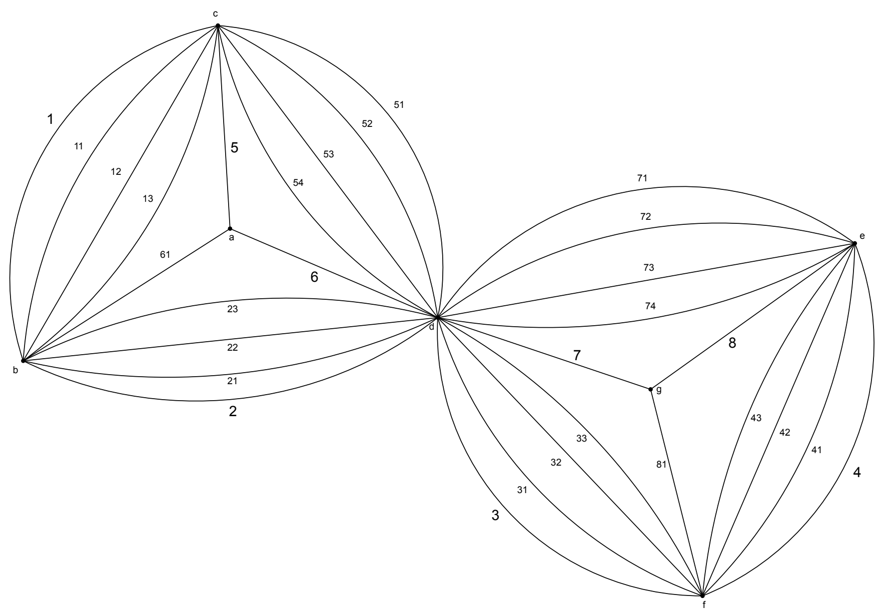

Figure 17: Graph G

We can see that the minimum degree the graph in Figure 17 is three. To show that the connectivity of the line graph is at least seven, we're going to need to choose various paths in $L(G)$ and show that between any two vertices in $V(L(G))$, there are at least seven distinct paths. We need to use Menger's Theorem, as stated below.

Theorem 3.1. *(Menger's Theorem) Let u and v be distinct vertices in the graph X. Then the maximum number of openly disjoint paths from u to v equals the minimum size of a set of vertices S such that u and v lie in distinct components of X/S.*

In our particular case, for the line graph, $L(G)$, this theorem tells us that if we can find seven vertex-independent paths from two vertices, then we will have the minimum vertex-connectivity. This will be the same as choosing any two edges in the graph of G and finding seven distinct paths. Theorem 2.1 stated that if a graph is m-line connected, the line graph is m connected. We will show that G is at least 7-line connected, which will give us that $\kappa(L(G)) \geq 7$.

Now, we have a few groups of edges we need to look at. We can think of different subsets of edges. Let $A1$ be the subset of edges consisting of 1, 11, 12, and 13, while $A2$ will consist of 4, 41, 42, and 43. Subset $B1$ will contain 2, 21, 22, 23, 51, 52, 53, and 54. In $B2$, we have 3, 31, 32, 33, 71, 72, 73, and 74. We choose subset $C1$ to have 5, 6, 61 and $C2$ to have 7, 8, and 81. Now, when we choose edge 1 in A and find disjoint paths between other vertices, we will not have to find disjoint paths with 11, 12, and 13.

We can choose an edge from each subset to represent the entire subset. We choose 1 from $A1$, 4 from $A2$, 2 from $B1$, 3 from $B2$, 5 from $C1$, and 7 from $C2$. First, we need to find paths between $A1$ and an edge from all of the subsets, including $A1$. We choose 11 to be another representative from $A1$. Most likely, there will be more than seven disjoint paths between each set of edges, but our minimum requirement is that $\kappa(L(G)) \geq 7$. We just need to focus on seven.

With edges 1 and 11, the disjoint paths are $1, 11$; $1, 12, 11$; $1, 13, 11$; $1, 2, 11$; $1, 21, 11$; $1, 22, 11$; and $1, 23, 11$. Next, we look at edges 1 and 4 and get $1, 2, 3, 4$; $1, 21, 31, 4$; $1, 22, 32, 4$; $1, 23, 33, 4$; $1, 51, 71, 4$; $1, 52, 72, 4$; and $1, 53, 73, 4$. For edges 1 and 2, disjoint paths are $1, 2$; $1, 51, 2$; $1, 52, 2$; $1, 53, 2$; $1, 54, 2$; $1, 11, 2$; and $1, 12, 2$. For edges 1 and 3, the paths are $1, 2, 3$; $1, 21, 3$; $1, 22, 3$; $1, 23, 3$; $1, 51, 3$; $1, 52, 3$; and $1, 53, 3$. So, we have paths for edges 1 and 5 as $1, 5$; $1, 11, 5$; $1, 12, 5$; $1, 13, 5$; $1, 21, 51, 5$; $1, 22, 52, 5$, and $1, 23, 53, 5$. Finally, disjoint paths between edges 1 and 7 are $1, 2, 7$; $1, 21, 7$; $1, 22, 7$; $1, 23, 7$; $1, 51, 7$; $1, 52, 7$; and $1, 53, 7$.

Now, we do not need to consider paths from $A2$ to other edges in other subsets. We already found paths from $A1$ to an edge from each subset, and since the graph is symmetric, finding paths from $A2$ to all the other edges in subsets would be similar to what we've already found. For instance, finding paths from $A1$ to $A1$ is like finding paths from $A2$ to $A2$. By finding paths from $A1$ to $B1$ and $B2$, we don't need to find paths from $A2$ to $B2$ and $B1$. Since we looked at $A1$ to $C1$ and $C2$, we won't need paths between $A2$ and $C2$ or $C1$.

So, we can move on to finding paths from an edge in $B1$ to other edges. We do not need to check $A1$ or $A2$, but we do need to look at $B1$. Choose 51 as another element of $B1$. Now, between 2 and 51, we have $2, 51$; $2, 21, 51$; $2, 22, 51$; $2, 23, 51$; $2, 52, 51$; $2, 53, 51$; and $2, 54, 51$. Next, for 2 and 3, the disjoint paths are $2, 3$; $2, 21, 3$; $2, 22, 3$; $2, 23, 3$; $2, 31, 3$; $2, 32, 3$; and $2, 33, 3$. Considering 2 and 5, we have $2, 1, 5$; $2, 11, 5$; $2, 12, 5$; $2, 13, 5$; $2, 51, 5$; $2, 52, 5$; and $2, 53, 5$. Finally, with 2 and 7, we get $2, 7$; $2, 21, 7$; $2, 22, 7$; $2, 23, 7$; $2, 51, 7$; $2, 52, 7$; and $2, 53, 7$.

Like with $A1$ and $A2$, we will not need to check $B2$ with any other edges.

So, we can move on to checking $C1$ with $C1$ and $C2$. Then, we will be finished with checking all disjoint paths between edges, since checking $C1$ checks $C2$ also. We choose 6 as another edge from $C1$. Between 5 and 6, the paths are $5, 6$; $5, 51, 6$; $5, 52, 6$; $5, 53, 6$; $5, 54, 6$; $5, 1, 2, 6$; and $5, 11, 21, 6$. For 5 and 7, we have $5, 6, 7$; $5, 51, 7$; $5, 52, 7$; $5, 53, 7$; $5, 54, 7$; $5, 1, 2, 7$; and $5, 11, 21, 7$.

We have found seven disjoint paths between each pair of edges in our graph, so the edge-connectivity of the graph is seven. Using Theorem 2.1, we can say that the vertex connectivity of the line graph of G is also seven. So, we have our condition for the graph.

Now, we can define a subgraph S of G as shown in Figure 18, and we can consider the graph of $G \setminus S$ in Figure 19.

We will now consider the previous lemma with this new graph. The graph has four components, so $\omega(G \setminus S) \geq 3$ is satisfied. We can now use Lemma 3.4 and go though each part of the lemma. Using the same notation, we can define the different

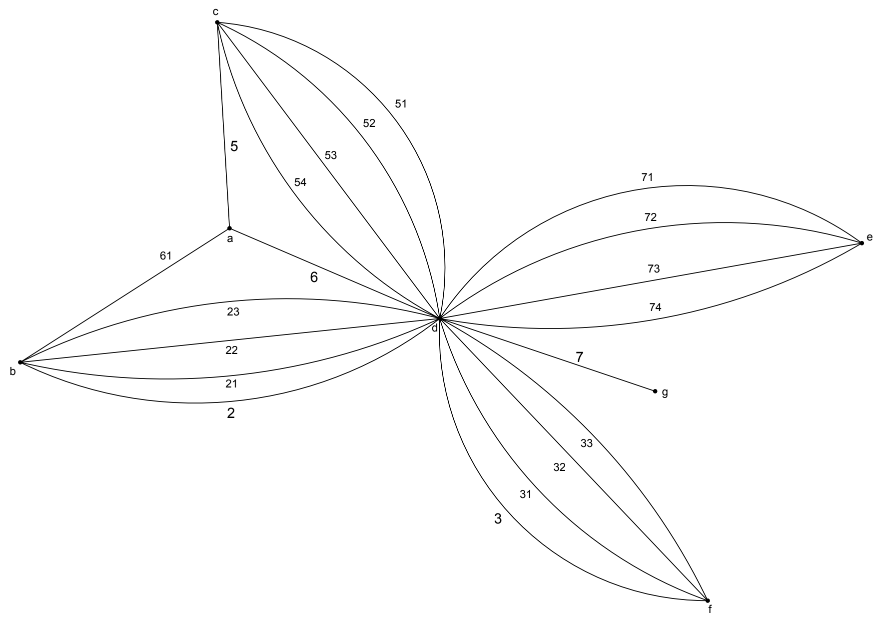

Figure 18: Graph S

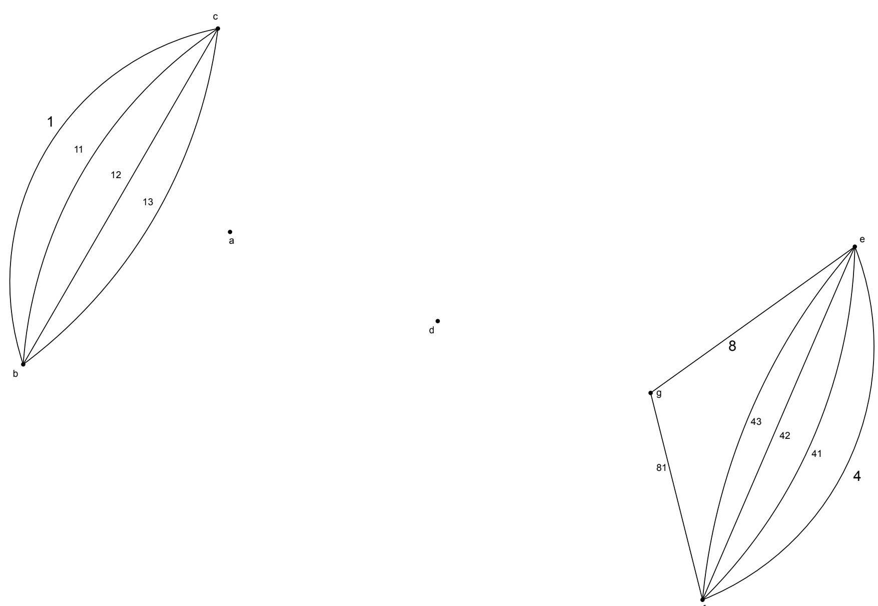

Figure 19: Graph $G \setminus S$

components, G_1, G_2, G_3, and G_4, in Figure 20. In this case, $r = 1, t = 2$, and $\omega = 4$.

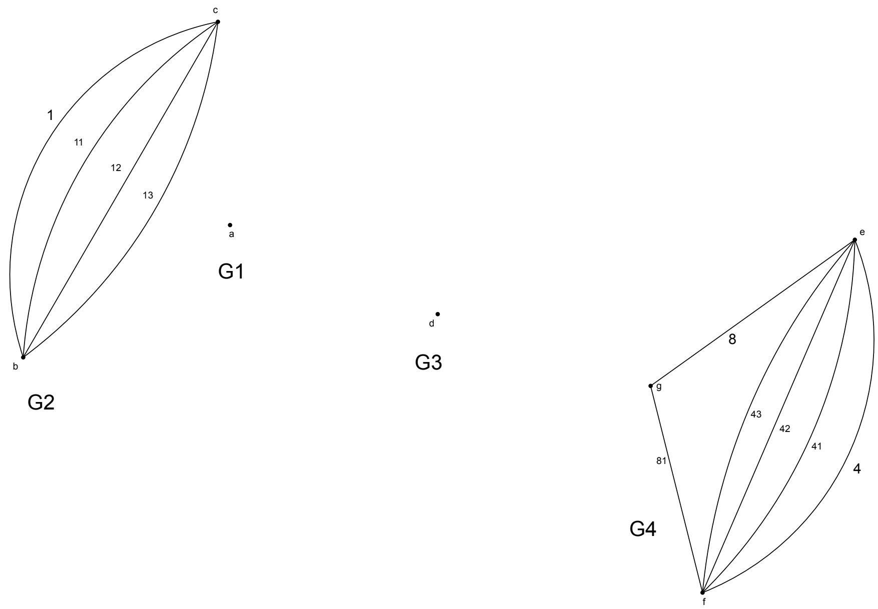

Figure 20: Components of $G \setminus S$

Now, we have the following details from Lemma 3.4. First, $m(G_1) = 3$ is clear, since we have three edges with one end-vertex in G_1. For Part 2, $m(G_2) \geq 6$ and $m(G_3) \geq 6$. In this case, $m(G_2) = 10$ and $m(G_3) = 18$. From Part 3, $m(G_4) \geq 4$, and we have $m(G_4) = 9$. Now, $\sum_2^3 m(G_i) \geq \sum_1^1 m(G_i) = m(G_1)$. Notice that $\sum_2^3 m(G_i) = m(G_2) + m(G_3) = 10 + 18 = 28$ and $m(G_1) = 3$. Finally, $\bigcup_1^4 M(G_i) \subseteq S$. Since all of the edges with one end vertex in each G_i were removed from the graph, and we removed all edges of S from the graph, we have that the set of edges of each G_i is a subset of the edges of S.

Next, we will consider the following lemma.

Lemma 3.5.

If S is a subset of $E(G)$, then

- $|S| \geq 2\omega(G \setminus S) - 1$, $if\ r = 1\ and\ \omega = 2$;
- $|S| \geq 2\omega(G \setminus S)$, $otherwise.$

Proof. Throughout the proof, we will just write ω for $\omega(G \setminus S)$. We will prove a few specific cases for r and ω, and then we will prove this for all other cases.

If $r = 0$, then we do not have a component consisting of a vertex of degree three. Also, we do not have any components from the second group, since this group depends on the original set of components. From Part 5 of Lemma 3.4, $\bigcup_1^\omega M(G_i) \subseteq S$. So, we can say that $|S| \geq |\bigcup_1^\omega M(G_i)|$. Since $m(G_i)$ is the cardinality of $M(G_i)$, $|\bigcup_1^\omega M(G_i)| \geq \frac{1}{2}\sum_1^\omega m(G_i)$, giving $|S| \geq \frac{1}{2}\sum_1^\omega m(G_i)$. Since $r = 0$ and $t = 0$, we can use Part 3 of Lemma 3.4 to say that $\frac{1}{2}\sum_1^\omega m(G_i) \geq \frac{1}{2}(4)\omega = 2\omega$. Therefore, we have $|S| \geq 2\omega$.

If $\omega = 0$, then we have deleted everything in our graph of $G - S$, and we have no components. So, $|S| \geq 0 = 2(0) = 2\omega$.

Next, when $\omega = 1$, then we can use $|S| \geq \frac{1}{2}\sum_1^\omega m(G_i)$ and say $|S| \geq \frac{1}{2}\sum_1^1 m(G_i) = \frac{1}{2}(4)(1) = 2(1) = 2\omega$ from Part 3 of Lemma 3.4. Therefore, $|S| \geq 2\omega$ for $\omega = 1$.

Now, if $r = 1$ and $\omega = 2$, then notice that we just have two components labeled G_1 and G_2, or G_r and G_ω, respectively. G_1 is a component consisting of a vertex of degree three and for G_2, the only thing we know is that it is a component that does not consist of a vertex of degree three. So, we can focus on G_1. Since it is a vertex of degree three, we know that S needed to have at least three edges in order for this component to have been created when we consider $G - S$. So, $|S| \geq 3$. Since we know $r = 1$ and $\omega = 2$, we get $2\omega - 1 = 2(2) - 1 = 3$, or that $|S| \geq 3 = 2\omega - 1$. Therefore, $|S| \geq 2\omega - 1$ when $r = 1$ and $\omega = 2$.

For the reminder of the proof, we will let $r \neq 0$ and $\omega \geq 3$. From Part 5 of Lemma 3.4, we can again say that $|S| \geq |\bigcup_1^\omega M(G_i)| \geq \frac{1}{2}\sum_1^\omega m(G_i)$.

Then, we can break up the $1, ..., \omega$ in the sum as $1, ..., r, r+1, ..., t, t+1, ..., \omega$, based on the way we defined the components of $G - S$. This gives us $\sum_1^\omega m(G_i) = \sum_1^r m(G_i) + \sum_{r+1}^t m(G_i) + \sum_{t+1}^\omega m(G_i)$. Combining this, we get

$|S| \geq \frac{1}{2}\sum_1^{\omega} m(G_i) = \frac{1}{2}(\sum_1^r m(G_i) + \sum_{r+1}^t m(G_i) + \sum_{t+1}^{\omega} m(G_i)) =$
$\frac{1}{2}\sum_1^r m(G_i) + \frac{1}{2}\sum_{r+1}^t m(G_i) + \frac{1}{2}\sum_{t+1}^{\omega} m(G_i)$.

Now, from Part 1 of Lemma 3.4, $m(G_i) = 3\ for\ 1 \leq i \leq r$, so $\sum_1^r m(G_i) = 3r$. Also, from Part 3 of Lemma 3.4, $m(G_i) \geq 4\ for\ t+1 \leq i \leq \omega$, which gives $\sum_{t+1}^{\omega} m(G_i) \geq 4(\omega - t)$. Our result is now $|S| \geq$
$\frac{1}{2}\sum_1^r m(G_i) + \frac{1}{2}\sum_{r+1}^t m(G_i) + \frac{1}{2}\sum_{t+1}^{\omega} m(G_i) \geq \frac{1}{2}(3r) + \frac{1}{2}(\sum_{r+1}^t m(G_i)) + \frac{1}{2}(4(\omega - t))$.

We need to consider two cases for $\sum_{r+1}^t m(G_i)$ to complete the proof.

For Case 1, we can assume $r \leq 2(t-r)$, or $(t-r) - \frac{1}{2}r \geq 0$. From Part 2 of Lemma 3.4 $m(G_i) \geq 6\ for\ r+1 \leq i \leq t$, so $\sum_{r+1}^t m(G_i) \geq 6(t-r)$. We now have $|S| \geq \frac{1}{2}(3r) + \frac{1}{2}(\sum_{r+1}^t m(G_i)) + \frac{1}{2}(4(\omega - t)) \geq \frac{1}{2}(3r) + \frac{1}{2}(6(t-r)) + \frac{1}{2}(4(\omega - t)) =$
$\frac{3}{2}r + 3t - 3r + 2\omega - 2t = -\frac{3}{2} + t + 2\omega = 2\omega + (t-r) - \frac{1}{2}r$. Using our assumption of $(t-r) - \frac{1}{2}r \geq 0$, $|S| \geq 2\omega + (t-r) - \frac{1}{2}r \geq 2\omega + 0 = 2\omega$. Therefore, $|S| \geq 2\omega$. For Case 2, we can say that $r > 2(t-r)$, or $r - 2(t-r) > 0$. Then, from Part 4 of Lemma 3.4, $\sum_{r+1}^t m(G_i) \geq \sum_1^r m(G_i)$. So, we can use what we know about $\sum_1^r m(G_i)$ to say that $\sum_{r+1}^t m(G_i) \geq \sum_1^r m(G_i) = 3r$. Now, we get
$|S| \geq \frac{1}{2}(3r) + \frac{1}{2}(\sum_{r+1}^t m(G_i)) + \frac{1}{2}(4(\omega - t)) \geq \frac{1}{2}3r + \frac{1}{2}3r + \frac{1}{2}4(\omega - t) =$
$3r + 2\omega - 2t = 2\omega + r - 2t + 2r = 2\omega + r - 2(t-r)$. Using the assumption, $r - 2(t-r) > 0$, we get $|S| \geq 2\omega + r - 2(t-r) > 2\omega + 0 = 2\omega$. Therefore, $|S| > 2\omega$.

Combining these two cases gives the result of $|S| \geq 2\omega$. □

Next, we will need to use a theorem from papers written by Nash-Williams [7] and Tutte [10].

Theorem 3.2. *In order that a finite graph G shall be decomposable into n connected factors it is necessary and sufficient that $|S| \geq n(\omega(G-S) - 1)$ for each subset S of the edge set $E(G)$.*

Using Lemma 3.5 and Theorem 3.2, we get the following corollary.

Corollary 3.1. For every x and y of edges of G, the subgraph $G - \{x, y\}$, or

$G - \{x\}$ if x and y have an end-vertex of degree 3 in common, can be decomposed into two connected factors F_1 and F_2.

From this corollary, we have the next lemma.

Lemma 3.6. Let x, y, and z be edges of G. If x and y are incident with a common vertex of degree 3, then there is a spanning closed trail containing y and z but not containing x; If x and y are not incident with a common vertex of degree 3, then there is a spanning closed trail containing z but not containing x or y.

This bring us to the last lemma needed for the proof of the main result.

Lemma 3.7. G is spanning trailable.

The proof for Lemma 3.7 uses Lemma 3.6.

Finally, the proof of Theorem 2.2 follows from using Lemma 3.1, Lemma 3.3, and Lemma 3.7. First, Lemma 3.7 gives that G is spanning trailable under the conditions that $\delta(G) \geq 3$ and $\kappa(L(G)) \geq 7$, or that we have a 7-connected line graph. From Lemma 3.3, G becomes dominating trailable. We can now use Lemma 3.1 to say that the line graph is hamiltonian-connected. Therefore, the line graph is hamiltonian-connected, and we have that every 7-connected line graph is hamiltonian-connected.

4 Conclusion

We have shown in the previous section that every 7-connected line graph is hamiltonian-connected. From 1991 until the year 2012, this was the main result that had been proven in the field of line graphs and hamiltonian paths. In July of 2012, we will get a more recent theorem, stating that every 5-connected line graph with minimum degree at least 6 is hamiltonian.

At the moment, the goal in this field is to prove Conjecture 2.1 and Conjecture 2.2. Since it was shown that these two conjectures are equivalent, only one will need to be proven for us to get both results.

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